Chapter 6

Three body decays

6.1 Muon decay

For a three body decay we have from eq. (4.1)

$$\begin{aligned}
d\Gamma =& \frac{1}{(2\pi)^5}\frac{1}{2M}\,|\mathcal{M}|^2\,\delta^4(P-p_1-p_2-p_3)\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\frac{d^3p_3}{2E_3}\\
=& \frac{1}{(2\pi)^5}\frac{1}{2M}\frac{d^3p_1}{2E_1}\int|\mathcal{M}|^2\,\delta^4(P-p_1-p_2-p_3)\frac{d^3p_2}{2E_2}\frac{d^3p_3}{2E_3}
\end{aligned} \tag{6.1}$$

6.1.1 Amplitude estimation

Since $\mathcal{M}$ is dimensionless, the amplitude averaged over spins for μ decay must be

$$|\mathcal{M}|^2 = CG_F^2 m_\mu^4\,. \tag{6.2}$$

We use

$$C = \frac{1}{2}(2\times 2\times 1\times 1) = 2 \tag{6.3}$$

The first factor is for the initial average and the factor are for the number of spin states of μ, e and the two neutrinos.

Consider first the integral

$$\begin{aligned}\int \delta^4(P-p_1-p_2)\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2} &= \int \delta(E-E_1-E_2)\delta^3(\mathbf{P}-\mathbf{p}_1-\mathbf{p}_2)\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\\ &= \int \delta(E-E_1-E_2)\frac{d^3p_2}{4E_1E_2}\int \delta^3(\mathbf{P}-\mathbf{p}_1-\mathbf{p}_2)d^3p_1\\ &= \int \delta(E-E_1-E_2)\frac{d^3p_2}{4E_1E_2}\int \delta^3(\mathbf{p}_1+\mathbf{p}_2-\mathbf{P})d^3p_1\\ &= \int \delta(E-E_1-E_2)\frac{d^3p_2}{4E_1E_2}\int \delta^3[\mathbf{p}_1-(\mathbf{P}-\mathbf{p}_2)]d^3p_1 \end{aligned} \tag{6.4}$$

using

$$\int \delta(x-x_0)dx = 1 \tag{6.5}$$

we have

$$\begin{aligned}\int \delta^4(P-p_1-p_2)\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2} &= \int \delta(E-E_1-E_2)\frac{d^3p_2}{4E_1E_2}\\ \int \delta^4(P-p_1-p_2)\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2} &= \int \delta(E-E_1-E_2)\frac{\mathbf{p}_2^2\,d|\mathbf{p}_2|d\Omega}{4E_1E_2}\end{aligned} \tag{6.6}$$

Since $|\mathbf{p}_1| = |\mathbf{p}_2|$ we have

$$\begin{aligned}E = E_1+E_2 =&(m_1^2+\mathbf{p}_1^2)^{1/2}+(m_2^2+\mathbf{p}_2^2)^{1/2}\\ =&(m_1^2+\mathbf{p}_2^2)^{1/2}+(m_2^2+\mathbf{p}_2^2)^{1/2}\end{aligned} \tag{6.7}$$

differentiating this equation with respect to $\mathbf{p}_2$

$$\begin{aligned}\frac{dE}{d|\mathbf{p}_2|} =&\frac{1}{2}\left(\frac{2|\mathbf{p}_2|}{(m_1^2+\mathbf{p}_2^2)^{1/2}}+\frac{2|\mathbf{p}_2|}{(m_2^2+\mathbf{p}_2^2)^{1/2}}\right)\\ =&|\mathbf{p}_2|\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\\ =&|\mathbf{p}_2|\left(\frac{E_1+E_2}{E_1E_2}\right)\end{aligned} \tag{6.8}$$

Therefore

$$d|\mathbf{p}_2| = \frac{dE}{|\mathbf{p}_2|}\left(\frac{E_1E_2}{E_1+E_2}\right) \tag{6.9}$$

replacing back in eq. (6.6)

$$\begin{aligned}\int \delta^4(P-p_1-p_2)\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2} &= \int \frac{dE}{|\mathbf{p}_2|}\,\delta(E-E_1-E_2)\frac{\mathbf{p}_2^2\,d\Omega}{4E_1E_2}\left(\frac{E_1E_2}{E_1+E_2}\right)\\ &= \int dE\,\delta(E-E_1-E_2)\frac{|\mathbf{p}_2|\,d\Omega}{4(E_1+E_2)}\\ &= \int \frac{|\mathbf{p}_2|}{4E}d\Omega \end{aligned}\tag{6.10}$$

For a relativistic particle $|\mathbf{p}_2| \approx E_2 = E/2$ and

$$\int \delta^4(P-p_1-p_2)\frac{d^3p_1}{E_1}\frac{d^3p_2}{E_2} = 2\pi \tag{6.11}$$

Applying this result to eq. (6.1) we have

$$\begin{aligned} d\Gamma =& \frac{1}{(2\pi)^5}\frac{1}{2M}\frac{d^3p_1}{2E_1}\int |\mathcal{M}|^2\,\delta^4(P-p_1-p_2-p_3)\frac{d^3p_2}{2E_2}\frac{d^3p_3}{2E_3}\\ =& \frac{1}{(2\pi)^5}\frac{1}{2M}\frac{d^3p_1}{8E_1}\,|\mathcal{M}|^2\int \delta^4(P-p_1-p_2-p_3)\frac{d^3p_2}{E_2}\frac{d^3p_3}{E_3}\\ =& \frac{1}{(2\pi)^5}\frac{1}{2M}\frac{d^3p_1}{8E_1}\,|\mathcal{M}|^2\,(2\pi)\\ =& \frac{G_F^2m_\mu^4}{8(2\pi)^4m_\mu E_1}\mathbf{p}_1^2d|\mathbf{p}_1|\int d\Omega\\ \approx& \frac{G_F^2m_\mu^3}{8(2\pi)^4E_1}E_1^2dE_1(4\pi)\\ \approx& \frac{G_F^2m_\mu^3}{4(2\pi)^3}E_1dE_1 \end{aligned}\tag{6.12}$$

As the maximum value of E_1 is $m_\mu/2$

$$\begin{aligned}\Gamma \approx& \frac{G_F^2m_\mu^3}{4(2\pi)^3}\int_0^{m_\mu/2} E_1dE_1\\ =& \frac{G_F^2m_\mu^3}{4(2\pi)^3}\frac{m_\mu^2}{8}\,, \end{aligned}\tag{6.13}$$

or

$$\Gamma = \frac{3}{4}\frac{G_F^2m_\mu^5}{192\pi^3}\,. \tag{6.14}$$

6.1.2 Amplitude calculation

The Standard Model Lagrangian includes

$$\begin{aligned}\mathcal{L} =& -\frac{\sqrt{2}g}{2}(\overline{\nu_e}_L\gamma^\mu e_L W^+_\mu + \bar{e}_L\gamma^\mu \nu_{eL} W^-_\mu + \overline{\nu_\mu}_L\gamma^\mu \mu_L W^+_\mu + \bar{\mu}_L\gamma^\mu \nu_{\mu L} W^-_\mu)\\ =& -\frac{g}{\sqrt{2}}(\overline{\nu_e}\gamma^\mu P_L e W^+_\mu + \bar{e}\gamma^\mu P_L \nu_e W^-_\mu + \overline{\nu_\mu}\gamma^\mu P_L \mu W^+_\mu + \bar{\mu}\gamma^\mu P_L \nu_\mu W^-_\mu)\\ =& -\frac{g}{2\sqrt{2}}(\overline{\nu_e}\gamma^\mu (1-\gamma_5) e W^+_\mu + \bar{e}\gamma^\mu (1-\gamma_5) \nu_e W^-_\mu + \overline{\nu_\mu}\gamma^\mu (1-\gamma_5) \mu W^+_\mu + \bar{\mu}\gamma^\mu (1-\gamma_5) \nu_\mu W^-_\mu)\end{aligned} \tag{6.15}$$

where

$$\begin{aligned}(\overline{\nu_e}\gamma^\mu P_L e W^+_\mu)^\dagger &= e^\dagger \gamma^\mu P_L{}^\dagger (\overline{\nu_e})^\dagger W^-_\mu = e^\dagger P_L \gamma^{\mu\dagger}\gamma^0 \nu_e W^-_\mu = e^\dagger \gamma^0\gamma^0\gamma^{\mu\dagger} P_R \gamma^0 \nu_e W^-_\mu\\ &= \bar{e}\gamma^0\gamma^{\mu\dagger}\gamma^0 P_L \nu_e W^-_\mu = \bar{e}\gamma^\mu P_L \nu_e W^-_\mu\end{aligned} \tag{6.16}$$

We can build the effective Lagrangian

Applying the Feynman rules to the diagram in fig. 6.1 we have the amplitude

$$\mathcal{M} = \frac{-ig^2}{8}\bar{u}_3\gamma_\mu(1-\gamma_5)u_1\left(\frac{-g^{\mu\nu}+q^\mu q^\nu/M_W^2}{q^2-M_W^2}\right)\bar{u}_4\gamma_\nu(1-\gamma_5)v_2 \tag{6.17}$$

where u (v) is for an incoming particle and $\bar{u}$ ($\bar{v}$) is for an ongoing particle (antiparticle).

The Dirac equations for spinors u and v are

$$\begin{aligned}(\not{p}-m)u &= 0 \qquad & (\not{p}+m)v &= 0\\ \bar{u}(\not{p}-m) &= 0 \qquad & \bar{v}(\not{p}+m) &= 0\,.\end{aligned} \tag{6.18}$$

In this way

$$\begin{aligned}\frac{1}{M_W^2}\gamma_\mu q^\mu(1-\gamma_5)u_1\bar{u}_4\gamma_\nu q^\nu(1-\gamma_5) =& \frac{1}{M_W^2}(1+\gamma_5)\not{q}u_1\bar{u}_4\not{q}(1-\gamma_5)\\ =& -\frac{m_\mu m_e}{M_W^2}(1+\gamma_5)u_1\bar{u}_4(1-\gamma_5)\end{aligned} \tag{6.19}$$

the term in $q^\mu q^\nu$ can be safely neglected. The term q^2 is m_μ^2 is small compared with m_W^2. Therefore

$$\begin{aligned}\mathcal{M} =& \frac{-ig^2}{8M_W^2}\bar{u}_3\gamma_\mu(1-\gamma_5)u_1\bar{u}_4\gamma^\mu(1-\gamma_5)v_2\\ =& \frac{-iG_F}{\sqrt{2}}\bar{u}_3\gamma_\mu(1-\gamma_5)u_1\bar{u}_4\gamma^\mu(1-\gamma_5)v_2\end{aligned} \tag{6.20}$$

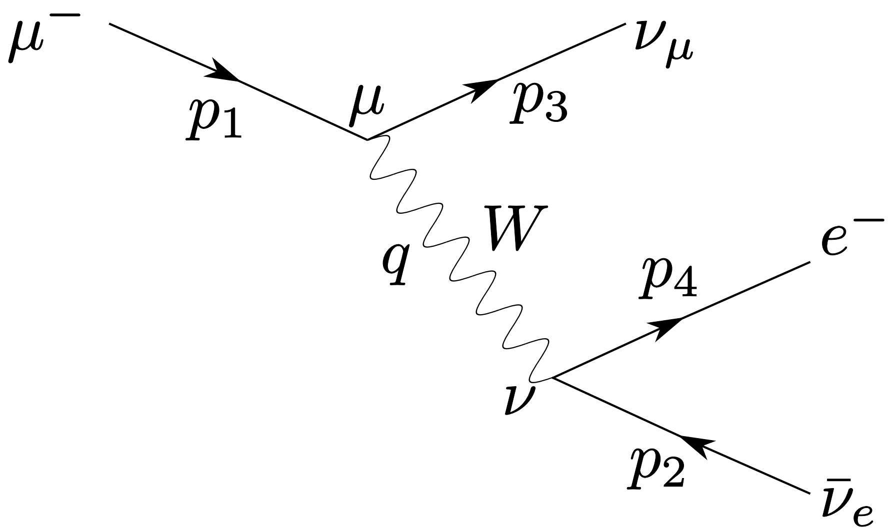

Figure 6.1: Tree level diagram for muon decay

$\mathcal{M}$ is a dimensionless scalar. The relevant coupling is

$$\frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}\,. \tag{6.21}$$

The conjugate is, following the same steps that in eq. (6.16)

$$\begin{aligned}\mathcal{M}^* =& \frac{ig^2}{8M_W^2}\left[\bar{u}_3\gamma_\mu(1-\gamma_5)u_1\right]^\dagger\left[\bar{u}_4\gamma^\mu(1-\gamma_5)v_2\right]^\dagger \\ \mathcal{M}^* =& \frac{ig^2}{8M_W^2}\left[\bar{u}_1\gamma_\mu(1-\gamma_5)u_3\right]\left[\bar{v}_2\gamma^\mu(1-\gamma_5)u_4\right]\,.\end{aligned} \tag{6.22}$$

Multiplying $\mathcal{M}$ and $\mathcal{M}^*$ we have

$$\begin{aligned}|\mathcal{M}|^2 =& \frac{g^4}{64M_W^4}\left[\bar{u}_3\gamma_\mu(1-\gamma_5)u_1\bar{u}_1\gamma_\nu(1-\gamma_5)u_3\right] \\ & \times\left[\bar{u}_4\gamma^\mu(1-\gamma_5)v_2\bar{v}_2\gamma^\nu(1-\gamma_5)u_4\right] \\ =& \frac{g^4}{64M_W^4}L_{\mu\nu}M^{\mu\nu}\end{aligned} \tag{6.23}$$

where

$$\begin{aligned}L_{\mu\nu} &= \left[\bar{u}_{3\alpha}\gamma_\mu^{\alpha\beta}(1-\gamma_5)_{\beta\gamma}u_1^\gamma\bar{u}_{1\delta}\gamma_\nu^{\delta\epsilon}(1-\gamma_5)_{\epsilon\eta}u_3^\eta\right] \\ M^{\mu\nu} &= \left[\bar{u}_4^\alpha\gamma_{\alpha\beta}^\mu(1-\gamma_5)^{\beta\gamma}v_{2\gamma}\bar{v}_2^\delta\gamma_{\delta\epsilon}^\nu(1-\gamma_5)^{\epsilon\eta}u_{4\eta}\right]\end{aligned} \tag{6.24}$$

$$
\begin{aligned}
L_{\mu\nu} &= \left[u_3^\eta \bar{u}_{3\alpha}\gamma_\mu^{\alpha\beta}(1-\gamma_5)_{\beta\gamma}u_1^\gamma \bar{u}_{1\delta}\gamma_\nu^{\delta\epsilon}(1-\gamma_5)_{\epsilon\eta}\right] \\
&= \left[(u_3\bar{u}_3)_{\eta\alpha}\gamma_\mu^{\alpha\beta}(1-\gamma_5)_{\beta\gamma}(u_1\bar{u}_1)_{\gamma\delta}\gamma_\nu^{\delta\epsilon}(1-\gamma_5)_{\epsilon\eta}\right] \\
&= \mathrm{Tr}\left[(u_3\bar{u}_3)\gamma_\mu(1-\gamma_5)(u_1\bar{u}_1)\gamma_\nu(1-\gamma_5)\right]
\end{aligned}
\tag{6.25}
$$

Using

$$
\sum_s u(p,s)\bar{u}(p,s) = (\not{p}+m) \qquad \sum_s v(p,s)\bar{v}(p,s) = (\not{p}-m)
\tag{6.26}
$$

$$
\begin{aligned}
\sum_s L_{\mu\nu} &= \mathrm{Tr}\left[(\not{p}_3)\gamma_\mu(1-\gamma_5)(\not{p}_1+m_\mu)\gamma_\nu(1-\gamma_5)\right] \\
&= p_3^\alpha\,\mathrm{Tr}\left[\gamma_\alpha\gamma_\mu(1-\gamma_5)(p_1^\beta\gamma_\beta+m_\mu)\gamma_\nu(1-\gamma_5)\right] \\
&= p_3^\alpha\,\mathrm{Tr}\left[(\gamma_\alpha\gamma_\mu-\gamma_\alpha\gamma_\mu\gamma_5)(p_1^\beta\gamma_\beta\gamma_\nu(1-\gamma_5)+m_\mu\gamma_\nu(1-\gamma_5))\right] \\
&= p_3^\alpha\,\mathrm{Tr}\Big[p_1^\beta\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu(1-\gamma_5)-p_1^\beta\gamma_\alpha\gamma_\mu\gamma_5\gamma_\beta\gamma_\nu(1-\gamma_5) \\
&\quad +m_\mu\gamma_\alpha\gamma_\mu\gamma_\nu(1-\gamma_5)-m_\mu\gamma_\alpha\gamma_\mu\gamma_5\gamma_\nu(1-\gamma_5)\Big] \\
&= p_3^\alpha\,\mathrm{Tr}\Big[p_1^\beta\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu-p_1^\beta\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_5-p_1^\beta\gamma_\alpha\gamma_\mu\gamma_5\gamma_\beta\gamma_\nu+p_1^\beta\gamma_\alpha\gamma_\mu\gamma_5\gamma_\beta\gamma_\nu\gamma_5 \\
&\quad +m_\mu\gamma_\alpha\gamma_\mu\gamma_\nu-m_\mu\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_5-m_\mu\gamma_\alpha\gamma_\mu\gamma_5\gamma_\nu+m_\mu\gamma_\alpha\gamma_\mu\gamma_5\gamma_\nu\gamma_5\Big]
\end{aligned}
\tag{6.27}
$$

as the trace of an odd number of γ–matrices is zero, we have

$$
\begin{aligned}
\sum_s L_{\mu\nu} &= p_3^\alpha p_1^\beta\,\mathrm{Tr}\left[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu-\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_5-\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_5+\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_5^2\right] \\
&= 2p_3^\alpha p_1^\beta\,\mathrm{Tr}\left[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu(1-\gamma_5)\right]
\end{aligned}
\tag{6.28}
$$

Similarly

$$
\sum_s M^{\mu\nu} = 2p_{4\delta}p_{2\epsilon}\,\mathrm{Tr}\left[\gamma^\delta\gamma^\mu\gamma^\epsilon\gamma^\nu(1-\gamma_5)\right]
\tag{6.29}
$$

substituting back in eq. (6.23) we have,

$$
\begin{aligned}
|\mathcal{M}|^2 =& \frac{g^4}{64M_W^4} 4p_3^\alpha p_1^\beta p_{4\delta} p_{2\epsilon} \,\mathrm{Tr}\left[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu(1-\gamma_5)\right] \mathrm{Tr}\left[\gamma^\delta\gamma^\mu\gamma^\epsilon\gamma^\nu(1-\gamma_5)\right] \\
=& \frac{g^4}{64M_W^4} 4p_3^\alpha p_1^\beta p_{4\delta} p_{2\epsilon}(64\delta_\delta^\alpha\delta_\epsilon^\beta) \\
=& \frac{g^4}{64M_W^4} 4\times 64(p_3\cdot p_4)(p_1\cdot p_2) \\
=& \frac{4g^4}{M_W^4}(p_3\cdot p_4)(p_1\cdot p_2) \\
=& 4\left(8\frac{g^2}{8M_W^2}\right)^2(p_3\cdot p_4)(p_1\cdot p_2) \\
=& 4\left(8\frac{G_F}{\sqrt{2}}\right)^2(p_3\cdot p_4)(p_1\cdot p_2) \\
=& 128\, G_F^2(p_3\cdot p_4)(p_1\cdot p_2) \\
=& 256\frac{G_F^2}{2}(p_3\cdot p_4)(p_1\cdot p_2)\,.
\end{aligned}
\tag{6.30}
$$

The demonstration of the used Tr×Tr identity can be found in Appendix B. of [7].

The spin–averaged differential decay width for $\mu^- \to \nu_\mu e^- \bar{\nu}_e$ is

$$
\begin{aligned}
d\Gamma =& \frac{1}{(2\pi)^5}\frac{1}{2E_1}\frac{d^3p_3}{2E_3}\left(\frac{1}{2}\sum|\mathcal{M}|^2\right)\delta^4(p_1-p_2-p_3-p_4)\frac{d^3p_2}{2E_2}\frac{d^3p_4}{2E_4} \\
=& \frac{1}{2E_1}\frac{1}{2}\sum|\mathcal{M}|^2\,\frac{1}{(2\pi)^5}\frac{d^3p_2}{8E_2}\delta^4(p_1-p_2-p_3-p_4)\frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4} \\
=& \frac{1}{2}\frac{4g^4}{M_W^4}\frac{1}{(2\pi)^5 2E_1}(p_1\cdot p_2)(p_3\cdot p_4)\frac{d^3p_2}{2E_2}\delta^4(p_1-p_2-p_3-p_4)\frac{d^3p_3}{2E_3}\frac{d^3p_4}{2E_4} \\
=& \frac{2g^4}{16(2\pi)^5 M_W^4 E_1 E_4} p_1^\beta p_4^\alpha d^3p_4 I_{\alpha\beta}
\end{aligned}
\tag{6.31}
$$

where the covariant integral $I_{\alpha\beta}$ on the neutrino momentum is

$$
I_{\alpha\beta} = \int p_{3\alpha}p_{2\beta}\delta^4(p-p_2-p_3)\frac{d^3p_2}{E_2}\frac{d^3p_3}{E_3}\,. \tag{6.32}
$$

The variable p in ec. (6.32) is defined as $p = p_1 - p_4 = p_2 + p_3$. Moreover

$$\begin{aligned} p^2 =& p_2^2 + p_3^2 + 2p_2 \cdot p_3 \\ =& m_{\nu_e}^2 + m_{\nu_\mu}^2 + 2p_2 \cdot p_3 \\ \approx& 2p_2 \cdot p_3 \\ g_{\alpha\beta} p^\alpha p^\beta =& 2 g_{\alpha\beta} p_3^\alpha p_2^\beta \\ p^\alpha p^\beta =& 2 p_3^\alpha p_2^\beta \,. \end{aligned} \tag{6.33}$$

$I_{\alpha\beta}$ must have the form

$$I_{\alpha\beta} = g_{\alpha\beta} A(p^2) + p_\alpha p_\beta B(p^2)\,. \tag{6.34}$$

Defining the itegral I as follows

$$I = \int \delta^4(p - p_2 - p_3) \frac{d^3p_2}{E_2} \frac{d^3p_3}{E_3}\,, \tag{6.35}$$

Since

$$\begin{aligned} m_\nu^2 \approx 0 =& E_\nu^2 - \mathbf{p}_\nu^2 \\ E_\nu^2 =& \mathbf{p}_\nu^2 \end{aligned} \tag{6.36}$$

and in addition the integral I is covariant, we choose to evaluate it in the rest frame of the two neutrinos $|\mathbf{p}_2| = |\mathbf{p}_3|$, which implies $E_2 = E_3$.

$$\begin{aligned} I =& \int \delta(E - E_2 - E_3)\delta^3(\mathbf{p} - \mathbf{p}_2 - \mathbf{p}_3) \frac{d^3p_2}{E_2} \frac{d^3p_3}{E_3} \\ =& \int \delta(E - E_2 - E_3) \frac{d^3p_2}{E_2 E_3} \underbrace{\int \delta^3(\mathbf{p} - \mathbf{p}_2 - \mathbf{p}_3) d^3p_3}_{=1} \\ =& \int \frac{\delta(E - 2E_2)}{E_2^2} \mathbf{p}_2^2 d|\mathbf{p}_2| d\Omega \\ =& \int \frac{\delta(E - 2E_2)}{E_2^2} E_2^2 dE_2 (4\pi) \\ =& 4\pi \int \delta\left(2\left(E_2 - \frac{E}{2}\right)\right) dE_2 \\ =& 4\pi \frac{1}{2} \int \delta\left(E_2 - \frac{E}{2}\right) dE_2 \\ =& 2\pi \end{aligned} \tag{6.37}$$

then multiplying (6.13) by $g^{\alpha\beta}$ and $p^\alpha p^\beta$ successively gives, using eq. (6.33)

$$g^{\alpha\beta} I_{\alpha\beta} = 4A + p^2 B = \int p_3 \cdot p_2 \delta^4(p - p_2 - p_3) \frac{d^3p_2}{E_2} \frac{d^3p_3}{E_3} = \frac{p^2}{2} I = \pi p^2$$

In order to compute $p^\alpha p^\beta I_{\alpha\beta}$, we make use of the fact that it is a Lorentz invariant quantity, so that we may evaluate it in any reference frame. In particular, we can evaluate it in the rest frame of the neutrinos involved in this process. This means that $p = p_2 + p_3 = (p^0, \mathbf{0})$ and $E_2 = E_3$

$$\begin{aligned}
p^\alpha p^\beta I_{\alpha\beta} &= p^2 A + p^4 B && (6.38)\\
&= p^\alpha p^\beta \int \frac{d^3p_2}{E_2} \frac{d^3p_3}{E_3} p_{3\alpha} p_{2\beta} \delta^4(p - p_2 - p_3) \\
&= \int \frac{d^3p_2}{E_2} \frac{d^3p_3}{E_3} E_3 p^0 E_2 p^0 \delta^4(p - p_2 - p_3) \\
&= (p^0)^2 \int d^3p_2 d^3p_3 \delta^4(p - p_2 - p_3) && (6.39)\\
&= (p^0)^2 \int d^3p_2 \delta(p^0 - 2E_2) \\
&= (p)^2 \int dE_2 E_2^2 d\Omega \frac{1}{2} \delta(\frac{p^0}{2} - E_2) = 4\pi \frac{p^2}{2} \left(\frac{p^2}{2}\right)^2 \\
&= \frac{\pi p^4}{2} && (6.40)
\end{aligned}$$

where the usual tricks have been used to simplify the integrals, using the delta function inside.

Therefore

$$A = \frac{p^2}{4}(\pi - B) \tag{6.41}$$

$$\begin{aligned}
\frac{p^4}{4}(\pi - B) + p^4 B &= \frac{\pi p^4}{2} \\
\frac{\pi}{4} - \frac{B}{4} + B &= \frac{\pi}{2} \\
\frac{3B}{4} &= \frac{\pi}{4} \\
B &= \frac{\pi}{3} && (6.42)
\end{aligned}$$

$$\begin{aligned} A =& \frac{p^2}{4}(\pi - \frac{\pi}{3}) \\ =& \frac{p^2}{4}(\frac{2\pi}{3}) \\ =& \frac{\pi p^2}{6} \end{aligned} \tag{6.43}$$

$$I_{\alpha\beta} = \frac{\pi}{6}\left(g_{\alpha\beta}p^2 + 2p_\alpha p_\beta\right) \ . \tag{6.44}$$

Substituting back in eq. (6.31) we have

$$\begin{aligned} d\Gamma =& \frac{2\pi g^4}{16 \times 6(2\pi)^5 M_W^4 E_1 E_4} p_1^\beta p_4^\alpha (g_{\alpha\beta}p^2 + 2p_\alpha p_\beta) d^3p_4 \\ d\Gamma =& \frac{2g^4}{16 \times 12(2\pi)^4 M_W^4 E_1 E_4}[(p_1 \cdot p_4)p^2 + 2(p \cdot p_1)(p \cdot p_4)]d^3p_4 \\ d\Gamma =& \frac{2g^4}{192(2\pi)^4 M_W^4 E_1 E_4}[(p_1 \cdot p_4)p^2 + 2(p \cdot p_1)(p \cdot p_4)]d^3p_4 \end{aligned} \tag{6.45}$$

For further evaluation we will use the rest frame of the decaying muon. In this frame the four–momentum are

$$\begin{aligned} p_1 =& (m_\mu, \mathbf{0}) \\ p_4 =& (E_4, \mathbf{p}_4) \\ p =& p_1 - p_4 = (m_\mu - E_4, -\mathbf{p}_4) \\ p^2 =& E^2 - \mathbf{p}^2 = m_\mu^2 - 2m_\mu E_4 + (E_4^2 - \mathbf{p}_4^2) = m_\mu^2 + m_e^2 - 2m_\mu E_4 \end{aligned} \tag{6.46}$$

Moreover

$$\begin{aligned} p_1 \cdot p_4 =& m_\mu E_4 \\ p \cdot p_1 =& m_\mu^2 - m_\mu E_4 \\ p \cdot p_4 =& m_\mu E_4 - E_4^2 + \mathbf{p}_4^2 = m_\mu E_4 - m_e^2 \\ p_4^2 = m_e^2 =& E_4^2 - \mathbf{p}_4^2 \Rightarrow \mathbf{p}_4^2 = E_4^2 - m_e^2 \\ |\mathbf{p}_4| =& (E_4^2 - m_e^2)^{1/2} \\ & \Rightarrow \frac{d|\mathbf{p}_4|}{dE_4} = \frac{1}{2}\frac{2E_4}{(E_4^2 - m_e^2)^{1/2}} = \frac{E_4}{|\mathbf{p}_4|} \\ & \Rightarrow d|\mathbf{p}_4| = \frac{E_4}{|\mathbf{p}_4|}dE_4 \\ d^3p_4 =& \mathbf{p}_4^2\, d|\mathbf{p}_4|\, d\Omega = |\mathbf{p}_4| E_4\, dE_4\, d\Omega \end{aligned} \tag{6.47}$$

Substituting back in eq. (6.45) we have

$$d\Gamma =\frac{2g^4}{192(2\pi)^4 M_W^4 m_\mu}|\mathbf{p}_4|\, dE_4\, d\Omega[(m_\mu^2+m_e^2-2m_\mu E_4)m_\mu E_4+2(m_\mu^2-m_\mu E_4)(m_\mu E_4-m_e^2)] \tag{6.48}$$

Neglecting electron mass we have $|\mathbf{p}_4|=E_4$, and

$$\begin{aligned} d\Gamma =&\frac{2g^4(4\pi)}{192(2\pi)^4 M_W^4 m_\mu}E_4\, dE_4[(m_\mu^2-2m_\mu E_4)m_\mu E_4+2(m_\mu^2-m_\mu E_4)m_\mu E_4]\\ =&\frac{2\times 2g^4}{192(2\pi)^3 M_W^4 m_\mu}m_\mu E_4^2[m_\mu^2-2m_\mu E_4+2m_\mu^2-2m_\mu E_4]\, dE_4\\ =&\frac{4g^4}{192(2\pi)^3 M_W^4}E_4^2\left[3m_\mu^2-4m_\mu E_4\right]\, dE_4\\ =&\frac{4g^4}{192(2\pi)^3 M_W^4}m_\mu^2 E_4^2\left[3-4\frac{E_4}{m_\mu}\right]\, dE_4\\ =&\frac{4g^4}{192(2\pi)^3 M_W^4}\frac{m_\mu^4}{4}\left(\frac{2E_4}{m_\mu}\right)^2\left[3-2\left(\frac{2E_4}{m_\mu}\right)\right]\frac{m_\mu}{2}\, d\left(\frac{2E_4}{m_\mu}\right)\\ =&\frac{4g^4}{192(2\pi)^3 M_W^4}\frac{m_\mu^5}{8}\left(\frac{2E_4}{m_\mu}\right)^2\left[3-2\left(\frac{2E_4}{m_\mu}\right)\right]\, d\left(\frac{2E_4}{m_\mu}\right) \end{aligned} \tag{6.49}$$

E_4 varies from 0 to $E_4^{\max}$ can be obtained from $(m_e=0)$

$$p_1-p_4=p_2+p_3\,. \tag{6.50}$$

The square of he factor on the left is

$$\begin{aligned} (p_1-p_4)^2 =&p_1^2+p_4^2-2p_1\cdot p_4\\ =&m_\mu^2+m_e^2-2p_1\cdot p_4\,. \end{aligned} \tag{6.51}$$

We have then using eqs. (6.47)(6.50)

$$\begin{aligned} 2p_1\cdot p_4 =&m_\mu^2+m_e^2-(p_1+p_4)^2\\ 2m_\mu E_4 =&m_\mu^2+m_e^2-(p_2+p_3)^2\\ \approx&m_\mu^2-(p_2+p_3)^2\,. \end{aligned} \tag{6.52}$$

$(p_2+p_3)^2$ is the invariant mass squared of the $\nu_\mu+\bar{\nu}_e$ system, which ranges from 0 to m_μ^2. For $(p_2+p_3)^2=m_\mu$ we have $E_4^{\min}=0$, while for $(p_2+p_3)^2=0$ we have $E_4^{\max}=m_\mu/2$. The missing integration on $d\Gamma$ is in the variable x such that

$$x=\frac{2E}{m_\mu}\,,\qquad x_{\min}=0\,,\qquad x_{\max}=\frac{2E_{\max}}{m_\mu}=1\,. \tag{6.53}$$

Therefore

$$
\begin{aligned}
\Gamma =& \frac{4g^4}{192(2\pi)^3 M_W^4}\frac{m_\mu^5}{8}\int_0^1 x^2[3-2x]\,dx \\
=& \frac{4g^4}{192(2\pi)^3 M_W^4}\frac{m_\mu^5}{8}\frac{1}{2} \\
=& \frac{g^4}{192\pi^3 8 M_W^4}\frac{m_\mu^5}{4} \\
=& \frac{g^4}{64 M_W^4}\frac{2}{192\pi^3}m_\mu^5 \\
=& \frac{G_F^2}{2}\frac{2}{192\pi^3}m_\mu^5 \\
=& \frac{G_F^2}{192\pi^3}m_\mu^5
\end{aligned}
\tag{6.54}
$$

Without neglect the electron mass we have

$$
\begin{aligned}
d\Gamma =& \frac{2g^4}{192(2\pi)^4 M_W^4 m_\mu}|\mathbf{p}_4|\,dE_4\,d\Omega[(m_\mu^2+m_e^2-2m_\mu E_4)m_\mu E_4 + 2(m_\mu^2 - m_\mu E_4)(m_\mu E_4 - m_e^2)] \\
=& \frac{4g^4}{192(2\pi)^3 M_W^4 m_\mu}\,dE_4(E_4^2-m_e^2)^{1/2}[m_\mu^3 E_4 + m_e^2 m_\mu E_4 - 2(m_\mu E_4)^2 \\
&+ 2m_\mu^3 E_4 - 2(m_\mu E_4)^2 - 2m_\mu^2 m_e^2 + 2m_\mu m_e^2 E_4] \\
d\Gamma =& \frac{4g^4}{192(2\pi)^3 M_W^4 m_\mu}\,dE_4(E_4^2-m_e^2)^{1/2}[3m_\mu^3 E_4 + 3m_e^2 m_\mu E_4 - 4(m_\mu E_4)^2 - 2m_\mu^2 m_e^2]
\end{aligned}
\tag{6.55}
$$

The decay width, in terms of $x = m_e/m_\mu$ is

$$
\begin{aligned}
\Gamma =& \frac{4g^4}{192(2\pi)^3 M_W^4 m_\mu}\int_{m_e}^{m_\mu(1+x^2)/2}(E_4^2-m_e^2)^{1/2}[(3m_\mu^2+3m_e^2-4m_\mu E_4)m_\mu E_4 - 2m_\mu^2 m_e^2]\,dE_4 \\
=& \frac{4g^4}{192(2\pi)^3 M_W^4 m_\mu}\frac{m_\mu^6}{16}I\left(x\right)\,, \qquad I(x) = 1 - 8x^2 - 24x^4\ln(x) + 8x^6 - x^8 \\
=& \frac{G_F^2 m_\mu^5}{192\pi^3}I\left(x\right) \\
=& \left(\frac{G_F}{\sqrt{2}}\right)^2\frac{m_\mu^5}{96\pi^3}I\left(x\right)\,,
\end{aligned}
\tag{6.56}
$$

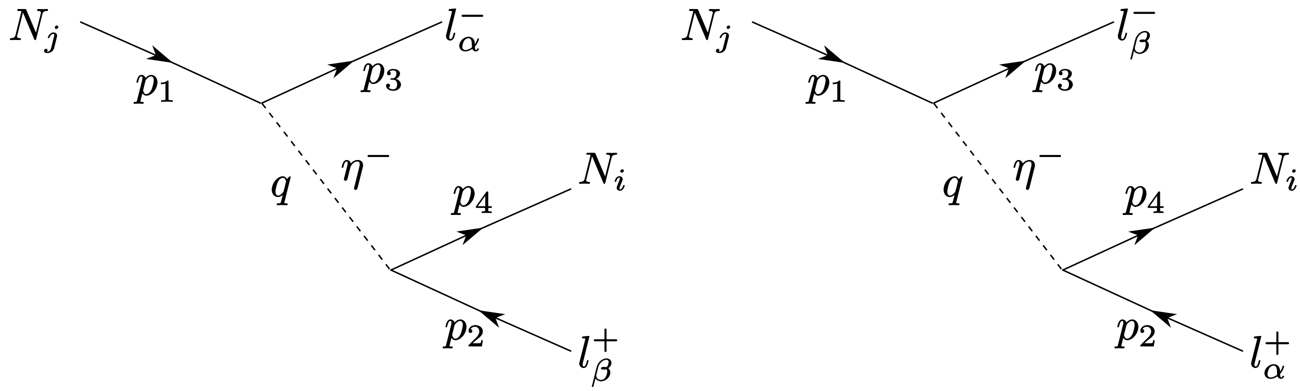

Figure 6.2: Tree level diagram for N_j decay

6.2 three body decays in radiative seesaw

We have the Lagrangian [14]

$$\begin{aligned}\mathcal{L} =&\epsilon_{ab}h_{\alpha j}\overline{N}_j P_L L^a_\alpha \eta^b + \text{h.c.}\\ =&h_{\alpha j}\overline{N}_j P_L L^1_\alpha \eta^2 - h_{\alpha j}\overline{N}_j P_L L^2_\alpha \eta^1 + \text{h.c.}\\ =&h_{\alpha j}\overline{N}_j P_L \nu_\alpha \eta^0 - h_{\alpha j}\overline{N}_j P_L l_\alpha \eta^+ + \text{h.c}\end{aligned} \tag{6.57}$$

where

$$(\overline{N}_j P_L l_\alpha \eta^+)^\dagger = l^\dagger_\alpha P_L \gamma^0 N_j \eta^- = \bar{l}_\alpha P_R \gamma^0 N_j \eta^- \tag{6.58}$$

Therefore

$$\begin{aligned}\mathcal{L} =&h_{\alpha j}\overline{N}_j P_L \nu_\alpha \eta^0 - h_{\alpha j}\overline{N}_j P_L l_\alpha \eta^+ + h^*_{\alpha j}\overline{\nu}_\alpha P_R N_j \eta^{*0} - h^*_{\alpha j}\bar{l}_\alpha P_R N_j \eta^-\\ =&\frac{1}{2}\left[h_{\alpha j}\overline{N}_j(1-\gamma_5)\nu_\alpha\eta^0 - h_{\alpha j}\overline{N}_j(1-\gamma_5)l_\alpha\eta^+ + h^*_{\alpha j}\overline{\nu}_\alpha(1+\gamma_5)N_j\eta^{*0} - h^*_{\alpha j}\bar{l}_\alpha(1+\gamma_5)N_j\eta^-\right]\end{aligned} \tag{6.59}$$

Applying Feynman rules to the diagram in fig.2 $N_j(p_1) \to l^-_\alpha(p_3)h^+$, $h^+ \to l^+_\beta(p_2) + N_i(p_4)$.

we have the amplitude

$$\begin{aligned}\mathcal{M} =&-ih_{\alpha j}\bar{u}_3(1-\gamma_5)u_1\left(\frac{1}{q^2-M^2_\eta}\right)h_{\beta i}\bar{u}_4(1-\gamma_5)v_2\\ &-ih_{\beta j}\bar{u}_3(1-\gamma_5)u_1\left(\frac{1}{q^2-M^2_\eta}\right)h_{\alpha i}\bar{u}_4(1-\gamma_5)v_2\\ \approx&-\frac{iH_{\alpha\beta ij}}{M^2_\eta}\bar{u}_3(1-\gamma_5)u_1\bar{u}_4(1-\gamma_5)v_2\end{aligned} \tag{6.60}$$

where

$$H_{\alpha\beta ij} = h_{\alpha j}h_{\beta i} + h_{\alpha i}h_{\beta j} \tag{6.61}$$

$$\begin{aligned}\mathcal{M}^* =& -\frac{iH_{\alpha\beta ij}}{M_\eta^2}[\bar{u}_3(1-\gamma_5)u_1]^\dagger[\bar{u}_4(1-\gamma_5)v_2]^\dagger \\ =& -\frac{iH_{\alpha\beta ij}}{M_\eta^2}[\bar{u}_1(1+\gamma_5)u_3][\bar{v}_2(1+\gamma_5)u_4]\,.\end{aligned} \tag{6.62}$$

Multiplying $\mathcal{M}$ and $\mathcal{M}^*$ we have

$$\begin{aligned}|\mathcal{M}|^2 =& \frac{H_{\alpha\beta ij}^2}{M_\eta^4}[\bar{u}_3^\alpha(1-\gamma_5)_{\alpha\beta}u_1^\beta\bar{u}_1^\gamma(1+\gamma_5)_{\gamma\delta}u_3^\delta][\bar{u}_4^\alpha(1-\gamma_5)_{\alpha\beta}v_2^\beta\bar{v}_2^\gamma(1+\gamma_5)_{\gamma\delta}u_4^\delta] \\ =& \frac{H_{\alpha\beta ij}^2}{M_\eta^4}[u_3^\delta\bar{u}_3^\alpha(1-\gamma_5)_{\alpha\beta}u_1^\beta\bar{u}_1^\gamma(1+\gamma_5)_{\gamma\delta}][u_4^\delta\bar{u}_4^\alpha(1-\gamma_5)_{\alpha\beta}v_2^\beta\bar{v}_2^\gamma(1+\gamma_5)_{\gamma\delta}] \\ =& \frac{H_{\alpha\beta ij}^2}{M_\eta^4}[(u_3\bar{u}_3)_{\delta\alpha}(1-\gamma_5)_{\alpha\beta}(u_1\bar{u}_1)_{\beta\gamma}(1+\gamma_5)_{\gamma\delta}][(u_4\bar{u}_4)_{\delta\alpha}(1-\gamma_5)_{\alpha\beta}(v_2\bar{v}_2)_{\beta\gamma}(1+\gamma_5)_{\gamma\delta}] \\ =& \frac{H_{\alpha\beta ij}^2}{M_\eta^4}\,\mathrm{Tr}[(u_3\bar{u}_3)(1-\gamma_5)(u_1\bar{u}_1)(1+\gamma_5)]\,\mathrm{Tr}[(u_4\bar{u}_4)(1-\gamma_5)(v_2\bar{v}_2)(1+\gamma_5)]\end{aligned} \tag{6.63}$$

Using eq. (6.26), and neglecting charged fermion masses

$$\begin{aligned}|\mathcal{M}|^2 =& \frac{H_{\alpha\beta ij}^2}{M_\eta^4}\,\mathrm{Tr}[\not{p}_3(1-\gamma_5)(\not{p}_1+M_j)(1+\gamma_5)]\,\mathrm{Tr}[(\not{p}_4+M_i)(1-\gamma_5)\not{p}_2(1+\gamma_5)] \\ =& \frac{H_{\alpha\beta ij}^2}{M_\eta^4}LM\end{aligned} \tag{6.64}$$

$$L = \mathrm{Tr}[(\not{p}_3 - \not{p}_3\gamma_5)(\not{p}_1 + \not{p}_1\gamma_5 + M_j + M_j\gamma_5] \tag{6.65}$$

$$\begin{aligned}L =& \mathrm{Tr}[\not{p}_3\not{p}_1 + \not{p}_3\not{p}_1\gamma_5 + M_j\not{p}_3 + M_j\not{p}_3\gamma_5 - \not{p}_3\gamma_5\not{p}_1 - \not{p}_3\gamma_5\not{p}_1\gamma_5 - \not{p}_3\gamma_5 M_j + M_j\gamma_5] \\ =& 2\,\mathrm{Tr}[\not{p}_3\not{p}_1] \\ =& 2p_3^\alpha p_1^\beta\,\mathrm{Tr}[\gamma_\alpha\gamma_\beta] \\ =& 8p_3^\alpha p_1^\beta g_{\alpha\beta} \\ =& 8(p_3\cdot p_1)\end{aligned} \tag{6.66}$$

Similarly

$$M = 8(p_4 \cdot p_2) \tag{6.67}$$

Therefore

$$\begin{aligned}|\mathcal{M}|^2 =& \frac{H^2_{\alpha\beta ij}}{M_\eta^4} 64(p_3 \cdot p_4)(p_1 \cdot p_2)\\ |\mathcal{M}|^2 =& \frac{H^2_{\alpha\beta ij}}{4M_\eta^4} 4 \times 64(p_3 \cdot p_4)(p_1 \cdot p_2)\end{aligned} \tag{6.68}$$

In this way, comparing with eq. (6.30), the results for the moun decay can be directly used after the replacements

$$\begin{aligned}\frac{g^4}{64M_W^4} &\to \frac{H^2_{\alpha\beta ij}}{4M_\eta^4}\\ \frac{g^4}{M_W^4} &\to \frac{16H^2_{\alpha\beta ij}}{M_\eta^4}\\ m_\mu &\to M_j\\ x = \frac{m_e}{m_\mu} &\to \frac{M_i}{M_j}\,.\end{aligned} \tag{6.69}$$

The decay width is according eq. (6.56)

$$\begin{aligned}\Gamma(N_j \to l^\mp_\alpha l^\pm_\beta N_i) =& \frac{16H^2_{\alpha\beta ij}}{M_\eta^4} \frac{4}{192(2\pi)^3 M_j} \frac{M_j^6}{16} I\,(x)\\ =& \frac{(h_{\alpha j}h_{\beta i} + h_{\alpha i}h_{\beta j})^2}{2M_\eta^4} \frac{M_j^5}{192\pi^3} I\,(x)\end{aligned} \tag{6.70}$$

where

$$I(x) = 1 - 8x^2 - 24x^4 \ln(x) + 8x^6 - x^8\,, \qquad x = \frac{M_i}{M_j}\,. \tag{6.71}$$

Similarly the decay through η^0 is

$$\Gamma(N_j \to \nu_\alpha \nu_\beta N_i) = \frac{(h_{\alpha j}h_{\beta i} + h_{\alpha i}h_{\beta j})^2}{2M_{\eta^0}^4} \frac{M_j^5}{192\pi^3} I\,(x) \tag{6.72}$$

In this way, for example for N_2

$$\begin{aligned}\sum_{\alpha}\Gamma(N_2\to l_\alpha^- l_\beta^+ N_1)=&\sum_{\alpha}\frac{h_{\alpha2}^2h_{\beta1}^2+h_{\alpha1}^2h_{\beta2}^2+2h_{\alpha2}h_{\alpha1}h_{\beta2}h_{\beta1}}{2M_\eta^4}\frac{M_2^5}{192\pi^3}I\left(x\right)\\ =&\frac{\mathbf{h}_2^2h_{\beta1}^2+\mathbf{h}_1^2h_{\beta2}^2+2\mathbf{h}_2\cdot\mathbf{h}_1h_{\beta2}h_{\beta1}}{2M_\eta^4}\frac{M_2^5}{192\pi^3}I\left(x\right)\end{aligned}\tag{6.73}$$

$$\begin{aligned}\sum_{\alpha\beta}\Gamma(N_2\to l_\alpha^- l_\beta^+ N_1)=&\frac{\mathbf{h}_2^2\mathbf{h}_1^2+\mathbf{h}_1^2\mathbf{h}_2^2+2(\mathbf{h}_2\cdot\mathbf{h}_1)^2}{2M_\eta^4}\frac{M_2^5}{192\pi^3}I\left(x\right)\\ =&\frac{\mathbf{h}_1^2\mathbf{h}_2^2+(\mathbf{h}_1\cdot\mathbf{h}_2)^2}{M_\eta^4}\frac{M_2^5}{192\pi^3}I\left(x\right)\end{aligned}\tag{6.74}$$

In general

$$\begin{aligned}\sum_{\alpha\beta}\Gamma(N_j\to l_\alpha^- l_\beta^+ N_i)=&\frac{\mathbf{h}_i^2\mathbf{h}_j^2+(\mathbf{h}_i\cdot\mathbf{h}_j)^2}{M_\eta^4}\frac{M_j^5}{192\pi^3}I\left(\frac{M_i}{M_j}\right)\\ \sum_{\alpha\beta}\Gamma(N_j\to \nu_\alpha\nu_\beta N_i)=&\frac{\mathbf{h}_i^2\mathbf{h}_j^2+(\mathbf{h}_i\cdot\mathbf{h}_j)^2}{M_{\eta^0}^4}\frac{M_j^5}{192\pi^3}I\left(\frac{M_i}{M_j}\right)\end{aligned}\tag{6.75}$$

For fix i and j

$$\frac{\sum_{\alpha\beta}\mathrm{Br}(N_j\to l_\alpha^- l_\beta^+ N_i)}{\sum_{\alpha\beta}\mathrm{Br}(N_j\to \nu_\alpha\nu_\beta N_i)}=\frac{M_{\eta^0}^4}{M_{\eta^\pm}^4}\tag{6.76}$$

while for

$$\begin{aligned}\frac{\sum_{\alpha\beta}\mathrm{Br}(N_3\to \nu_\alpha\nu_\beta N_2)}{\sum_{\alpha\beta}\mathrm{Br}(N_3\to l_\alpha^- l_\beta^+ N_1)}\approx&\frac{\mathbf{h}_2^2\mathbf{h}_3^2+(\mathbf{h}_2\cdot\mathbf{h}_3)^2}{\mathbf{h}_1^2\mathbf{h}_3^2+(\mathbf{h}_1\cdot\mathbf{h}_3)^2}\frac{M_{\eta^\pm}^4}{M_{\eta^0}^4}I(M_2/M_3)\\ \frac{\sum_{\alpha\beta}\mathrm{Br}(N_3\to l_\alpha^- l_\beta^+ N_2)}{\sum_{\alpha\beta}\mathrm{Br}(N_3\to l_\alpha^- l_\beta^+ N_1)}\approx&\frac{\mathbf{h}_2^2\mathbf{h}_3^2+(\mathbf{h}_2\cdot\mathbf{h}_3)^2}{\mathbf{h}_1^2\mathbf{h}_3^2+(\mathbf{h}_1\cdot\mathbf{h}_3)^2}I(M_2/M_3)\end{aligned}\tag{6.77}$$

For N_2 the total decay width is

$$\Gamma_{\mathrm{tot}}(N_2)=\left[\mathbf{h}_1^2\mathbf{h}_2^2+(\mathbf{h}_1\cdot\mathbf{h}_2)^2\right]\frac{M_2^5}{192\pi^3}I\left(\frac{M_1}{M_2}\right)\left[\frac{1}{M_{\eta^\pm}^4}+\frac{1}{M_{\eta^0}^4}\right]\tag{6.78}$$

And the individual branchings through $\eta^{\pm}$ given by eq. (6.70).

For N_3 we have several possibilities for signals with charged leptons. The cleanest one is when N_3 decay only through $\eta^{\pm}$ through an intermediate N_2.

The branching of N_3 to two charged leptons plus missing energy is either

$$\mathrm{Br}(N_3 \to l_\alpha^{\pm} l_\beta^{\mp} N_1) \tag{6.79}$$

where the N_3 is reconstructed, or

$$\mathrm{Br}(N_3 \underbrace{\to}_{\eta^0} l_\alpha^{\pm} l_\beta^{\mp} N_1) = \mathrm{Br}(N_3 \to \nu_\alpha \nu_\beta N_2) \times \mathrm{Br}(N_2 \to l_\alpha^{\pm} l_\beta^{\mp} N_1) \tag{6.80}$$

that seem to be very difficult to reconstruct. This also seem to be an irreducible background for

$$\mathrm{Br}(N_2 \to l_\alpha^{\pm} l_\beta^{\mp} N_1) \tag{6.81}$$

To get rid of processes like the one in eq. (6.80) must be $\mathrm{Br}(N_3 \to \nu_\alpha \nu_\beta N_2)$ is suppressed. This happens if

- $I(M_2/M3) \ll 1$. In this case the mutilepton signal for N_3 is also suppressed. Clearly this happens for $M_2 \approx M_3$ as $I(x)$ is a sharpest function which controls the kinematical suppression. We show below for an specific point that even for $M_3 - M_2 \approx 20\,\mathrm{GeV}$, we can have the Branching in eq. (6.79) sufficiently large.

- $M_{\eta^{\pm}} \ll M_{\eta^0}$

In appendix 6.A, it is shown a full set of yukawas consistent with neutrino physics. For this solution

$$\frac{\mathrm{Br}(\eta^+ \to N_3)}{\mathrm{Br}(\eta^+ \to N_1)} \approx 0.61 \qquad \frac{\mathrm{Br}(\eta^+ \to N_2)}{\mathrm{Br}(\eta^+ \to N_1)} \approx 0.37$$
$$\mathrm{Br}(\eta^+ \to N_1) \approx 0.51 \qquad \mathrm{Br}(\eta^+ \to N_2) \approx 0.19 \qquad \mathrm{Br}(\eta^+ \to N_3) \approx 0.30 \tag{6.82}$$

Below we estimate the branchings to $N_3 \to l_\alpha^- l_\beta^+ N_1$ or $N_3 \to \nu_\alpha \nu_\beta N_2 \to \nu_\alpha \nu_\beta l_\alpha^- l_\beta^+ N_1$. For this we need the Branchings for $N_2 \to l_\alpha^- l_\beta^+ N_1$ compared with Branching to $N_2 \to \nu_\alpha \nu_\beta N_1$. In general this is From this, the visible decays are using eq. (6.76)

$$\frac{\sum_{\alpha\beta} \mathrm{Br}(N_2 \to l_\alpha^- l_\beta^+ N_1)}{\sum_{\alpha\beta} \mathrm{Br}(N_2 \to \nu_\alpha \nu_\beta N_1)} \approx 0.758 \Rightarrow \qquad \sum_{\alpha\beta} \mathrm{Br}(N_2 \to l_\alpha^- l_\beta^+ N_1) = 0.431 \tag{6.83}$$

On the other hand the chanels for N_3 are $N_3 \to l_\alpha^- l_\beta^+ N_1$, $N_3 \to \nu_\alpha \nu_\beta N_1$, $N_3 \to l_\alpha^- l_\beta^+ N_2$, and $N_3 \to \nu_\alpha \nu_\beta N_2$. From eqs. (6.76) (6.77)

$$\frac{\sum_{\alpha\beta} \mathrm{Br}(N_3 \to \nu_\alpha \nu_\beta N_2)}{\sum_{\alpha\beta} \mathrm{Br}(N_3 \to l_\alpha^- l_\beta^+ N_1)} \approx 0.0812 \qquad \frac{\sum_{\alpha\beta} \mathrm{Br}(N_3 \to l_\alpha^- l_\beta^+ N_2)}{\sum_{\alpha\beta} \mathrm{Br}(N_3 \to l_\alpha^- l_\beta^+ N_1)} \approx 0.0615$$
$$\frac{\sum_{\alpha\beta} \mathrm{Br}(N_3 \to \nu_\alpha \nu_\beta N_1)}{\sum_{\alpha\beta} \mathrm{Br}(N_3 \to l_\alpha^- l_\beta^+ N_1)} \approx 1.320 \tag{6.84}$$

$$\begin{aligned}
\sum_{\alpha\beta} \mathrm{Br}(N_3 \to l_\alpha^- l_\beta^+ N_1) &\approx \frac{1}{1 + 0.0812 + 0.0615 + 1.320} = 0.406 \\
\sum_{\alpha\beta} \mathrm{Br}(N_3 \to \nu_\alpha \nu_\beta N_1) &\approx 0.536 \\
\sum_{\alpha\beta} \mathrm{Br}(N_3 \to \nu_\alpha \nu_\beta N_2) &\approx 0.030 \\
\sum_{\alpha\beta} \mathrm{Br}(N_3 \to l_\alpha^- l_\beta^+ N_2) &\approx 0.025
\end{aligned} \tag{6.85}$$

The expected background for $N_{2,3} \to l_\alpha^- l_\beta^+ N_1$ is

$$\sum_{\alpha\beta} \mathrm{Br}(N_3 \to \nu_\alpha \nu_\beta N_2) \times \sum_{\alpha\beta} \mathrm{Br}(N_2 \to l_\alpha^- l_\beta^+ N_1) \approx 0.030 \times 0.431 = 0.013 \tag{6.86}$$

We have that

$$\Gamma_{tot}(N_2) = \left[\mathbf{h}_1^2\mathbf{h}_2^2 + (\mathbf{h}_1\cdot\mathbf{h}_2)^2\right]\frac{M_2^5}{192\pi^3}I\left(\frac{M_1}{M_2}\right)\left[\frac{1}{M_{\eta^\pm}^4}+\frac{1}{M_{\eta^0}^4}\right] \tag{6.87}$$

$$\Gamma_{vis}(N_2\to N_1) \equiv \sum_{\alpha\beta}\Gamma(N_2\to l_\alpha^- l_\beta^+ N_1) \tag{6.88}$$

$$= \frac{\mathbf{h}_1^2\mathbf{h}_2^2 + (\mathbf{h}_1\cdot\mathbf{h}_2)^2}{M_{\eta^\pm}^4}\frac{M_2^5}{192\pi^3}I\left(\frac{M_1}{M_2}\right) \tag{6.89}$$

$$\Gamma_{vis}(N_3\to N_1) \equiv \sum_{\alpha\beta}\Gamma(N_3\to l_\alpha^- l_\beta^+ N_1) \tag{6.90}$$

$$= \frac{\mathbf{h}_1^2\mathbf{h}_3^2 + (\mathbf{h}_1\cdot\mathbf{h}_3)^2}{M_{\eta^\pm}^4}\frac{M_3^5}{192\pi^3}I\left(\frac{M_1}{M_3}\right) \tag{6.91}$$

$$\Gamma_{invis}(N_3\to N_2) \equiv \sum_{\alpha\beta}\Gamma(N_3\to \nu_\alpha\nu_\beta N_2) \tag{6.92}$$

$$= \frac{\mathbf{h}_2^2\mathbf{h}_3^2 + (\mathbf{h}_2\cdot\mathbf{h}_3)^2}{M_{\eta^0}^4}\frac{M_3^5}{192\pi^3}I\left(\frac{M_2}{M_3}\right). \tag{6.93}$$

From above equations we can obtain the following observable:

$$\frac{\mathrm{Br}_{invis}(N_3\to N_2)\times\mathrm{Br}_{vis}(N_2\to N_1)}{\mathrm{Br}_{vis}(N_3\to N_1)} \tag{6.94}$$

$$= \frac{\frac{\mathbf{h}_2^2\mathbf{h}_3^2+(\mathbf{h}_2\cdot\mathbf{h}_3)^2}{M_{\eta^0}^4}\frac{M_3^5}{192\pi^3}I\left(\frac{M_2}{M_3}\right)\times\frac{\mathbf{h}_1^2\mathbf{h}_2^2+(\mathbf{h}_1\cdot\mathbf{h}_2)^2}{M_{\eta^\pm}^4}\frac{M_2^5}{192\pi^3}I\left(\frac{M_1}{M_2}\right)}{\frac{\mathbf{h}_1^2\mathbf{h}_3^2+(\mathbf{h}_1\cdot\mathbf{h}_3)^2}{M_{\eta^\pm}^4}\frac{M_3^5}{192\pi^3}I\left(\frac{M_1}{M_3}\right)\Gamma_{tot}(N_2)} \tag{6.95}$$

$$= \frac{\mathbf{h}_2^2\mathbf{h}_3^2 + (\mathbf{h}_2\cdot\mathbf{h}_3)^2}{\mathbf{h}_1^2\mathbf{h}_3^2 + (\mathbf{h}_1\cdot\mathbf{h}_3)^2}I\left(\frac{M_2}{M_3}\right)\frac{1}{M_{\eta^0}^4\left[\frac{1}{M_{\eta^0}^4}+\frac{1}{M_{\eta^\pm}^4}\right]} \tag{6.96}$$

$$= \frac{\mathbf{h}_2^2\mathbf{h}_3^2 + (\mathbf{h}_2\cdot\mathbf{h}_3)^2}{\mathbf{h}_1^2\mathbf{h}_3^2 + (\mathbf{h}_1\cdot\mathbf{h}_3)^2}I\left(\frac{M_2}{M_3}\right)\frac{1}{\left[1+\frac{M_{\eta^0}^4}{M_{\eta^\pm}^4}\right]} \tag{6.97}$$

6.A Sample point

```
write(32,*) (h(i,1),i=1,3),(h(i,2),i=1,3),(h(i,3),i=1,3)
```

```
-0.00188878597  0.000780236776  0.000248251388
-0.000352494763 -0.000180683976 -0.00122443053
 0.000392272581  0.00120920029 -0.0012245638
```

So that

$$\begin{aligned} \mathbf{h}_1^2 &\approx 4.238\times 10^{-6} & \mathbf{h}_2^2 &\approx 1.656\times 10^{-6} \\ \mathbf{h}_3^2 &\approx 3.116\times 10^{-6} & \mathbf{h}_1\cdot\mathbf{h}_2 &\approx 2.208\times 10^{-7} \\ \mathbf{h}_1\cdot\mathbf{h}_3 &\approx 1.015\times 10^{-7} & \mathbf{h}_2\cdot\mathbf{h}_3 &\approx 1.143\times 10^{-6} \end{aligned} \tag{6.98}$$

$$\begin{aligned} \mathbf{h}_1^2\mathbf{h}_2^2 + (\mathbf{h}_1\cdot\mathbf{h}_2)^2 &\approx 7.067\times 10^{-12} & \mathbf{h}_1^2\mathbf{h}_3^2 + (\mathbf{h}_1\cdot\mathbf{h}_3)^2 &\approx 1.321\times 10^{-11} \\ \mathbf{h}_2^2\mathbf{h}_3^2 + (\mathbf{h}_2\cdot\mathbf{h}_3)^2 &\approx 6.465\times 10^{-12} \end{aligned} \tag{6.99}$$

The spectrum consistent with neutrino data is

$$\begin{aligned} M_1 &\approx 6.16918656\,\text{KeV} & M_2 &\approx 22.8695451\,\text{GeV} & M_3 &\approx 43.126911\,\text{GeV} \\ M_{\eta^0} &\approx 139.1382\,\text{GeV} & M_{\eta^\pm} &\approx 149.1382\,\text{GeV} \end{aligned} \tag{6.100}$$

$$I(M_1/M_3) \approx 1 \qquad\qquad I(M_2/M_3) \approx 0.126 \tag{6.101}$$

6.B Preliminary discussion

One interesting possibility in view of the large invisible direct decay, like $N_3 \to \nu_\alpha \nu_\beta N_1$, is to get the observables from the missing plus one energetic lepton (coming from η^+) signal. May be decays like

$$\begin{aligned} \eta^+ &\to l_\alpha^+ N_3 \to l_\alpha^+ \not{E}_T \\ \eta^+ &\to l_\alpha^+ N_2 \to l_\alpha^+ \not{E}_T \end{aligned} \tag{6.102}$$

Once $\eta^0_{R,I}$, or $\eta^\pm$ are produced the full list of signals is: For $\eta^\pm$ production. The decay to N_j is

$$\Gamma(\eta^+ \to l_\alpha^+ N_j) = \frac{3h_{\alpha j}^2}{16\pi M_\eta}\lambda^{1/2}\left(M_\eta^2, M_j^2, m_\alpha^2\right)\left(1 - \frac{M_j^2 + m_\alpha^2}{M_\eta^2}\right) \tag{6.103}$$

$$\sum_\alpha \Gamma(\eta^+ \to l_\alpha^+ N_j) = \frac{3\mathbf{h}_j^2}{16\pi M_\eta}\lambda^{1/2}\left(M_\eta^2, M_j^2, m_\alpha^2\right)\left(1 - \frac{M_j^2 + m_\alpha^2}{M_\eta^2}\right) \tag{6.104}$$

with

$$\lambda^{1/2}\left(M_\eta^2, M_j^2, m_\alpha^2\right) = \left[\left(M_\eta^2 + M_j^2 - m_\alpha^2\right)^2 - 4M_\eta^2 M_j^2\right]^{1/2} \tag{6.105}$$

Neglecting m_α with respect to $N_{2,3}$, we have for $j = 2, 3$

$$\begin{aligned}\lambda^{1/2}\left(M_\eta^2, M_j^2, m_\alpha^2\right) \approx& M_\eta^2 \left[\left(1 + \frac{M_j^2}{M_\eta^2}\right)^2 - \frac{4M_j^2}{M_\eta^2}\right]^{1/2} \\ \approx& M_\eta^2 \left[1 + 2\frac{M_j^2}{M_\eta^2} - \frac{4M_j^2}{M_\eta^2}\right]^{1/2} \\ \approx& M_\eta^2 \left[1 - 2\frac{M_j^2}{M_\eta^2}\right]^{1/2} \\ \approx& M_\eta^2 \left[1 - \frac{M_j^2}{M_\eta^2}\right]\end{aligned} \tag{6.106}$$

Therefore

$$\begin{aligned}\sum_\alpha \Gamma(\eta^+ \to l_\alpha^+ N_j) \approx& \frac{3\mathbf{h}_j^2 M_\eta}{16\pi} \times \begin{cases}\left(1 - \frac{M_j^2}{M_\eta^2}\right)^2 & j = 2, 3 \\ 1 & j = 1\end{cases} \\ \approx& \frac{3\mathbf{h}_j^2 M_\eta}{16\pi} \times \begin{cases}\left(1 - 2\frac{M_j^2}{M_\eta^2}\right) & j = 2, 3 \\ 1 & j = 1\end{cases}\end{aligned} \tag{6.107}$$

In this way

$$\begin{aligned}\Gamma_{\rm tor}(\eta^+) =& \sum_{\alpha j} \Gamma(\eta^+ \to l_\alpha^+ N_j) \\ \approx& \frac{3M_\eta}{16\pi}\left[\mathbf{h}_1^2 + \mathbf{h}_2^2\left(1 - 2\frac{M_2^2}{M_\eta^2}\right) + \mathbf{h}_3^2\left(1 - 2\frac{M_3^2}{M_\eta^2}\right)\right]\end{aligned} \tag{6.108}$$

$$
\begin{aligned}
\frac{\mathrm{Br}(\eta^+ \to N_j)}{\mathrm{Br}(\eta^+ \to N_i)} =& \frac{\sum_\alpha \Gamma(\eta^+ \to l_\alpha^+ N_j)}{\sum_\alpha \Gamma(\eta^+ \to l_\alpha^+ N_i)} \\
\approx& \frac{\mathbf{h}_j^2}{\mathbf{h}_i^2} \frac{1 - 2M_j^2/M_\eta^2}{1 - 2M_i^2/M_\eta^2} \\
\approx& \frac{\mathbf{h}_j^2}{\mathbf{h}_i^2} (1 - 2M_j^2/M_\eta^2)(1 - 2M_i^2/M_\eta^2)^{-1} \\
\approx& \frac{\mathbf{h}_j^2}{\mathbf{h}_i^2} (1 - 2M_j^2/M_\eta^2)(1 + 2M_i^2/M_\eta^2) \\
\approx& \frac{\mathbf{h}_j^2}{\mathbf{h}_i^2} \left[1 - 2\left(\frac{M_j^2 - M_i^2}{M_\eta^2}\right)\right]
\end{aligned} \tag{6.109}
$$

For three branchings we should have

$$
\begin{aligned}
a + b + c &= 1 \\
1 + \frac{b}{a} + \frac{c}{a} &= \frac{1}{a} \\
a &= \frac{1}{1 + b/a + c/a}
\end{aligned} \tag{6.110}
$$

In this way

$$
\mathrm{Br}(\eta^+ \to N_1) = \frac{1}{1 + \frac{\mathrm{Br}(\eta^+ \to N_3)}{\mathrm{Br}(\eta^+ \to N_1)} + \frac{\mathrm{Br}(\eta^+ \to N_3)}{\mathrm{Br}(\eta^+ \to N_1)}} \tag{6.111}
$$

From eq.

$$
\frac{\mathrm{Br}\,(N_3 \to N_1)}{\mathrm{Br}(N_3 \underbrace{\to}_{\eta^\pm} N_2)} = \tag{6.112}
$$