Chapter 3

S–matrix

We will use the S–matrix formulation to obtain the decay rates and cross section formulas. The relevant and properly normalized of the S–matriz

3.1 The S–matrix

In the Scrödinger picture, the state of a system evolves with time

$$\begin{aligned} |a,t\rangle =&U(t,t_i)|a,t_i\rangle \\ |a,t\rangle =&U(t,t_i)|a\rangle \\ |a,t\rangle =&e^{-iH(t-t_i)}|a\rangle\,, \end{aligned} \tag{3.1}$$

where $|a,t_i\rangle$, at an initial time t_i, is an eigenstate of a set of conmuting operators, and is denoted simply $|a\rangle$. Similarly $|b\rangle = |b\rangle(t_f)$.

We have then

$$\begin{aligned} \langle b,t_f|a,t_f\rangle =&\langle b|a,t_f\rangle \\ =&\langle b|e^{-iH(t_f-t_i)}|a,t_i\rangle \\ =&\langle b|e^{-iH(t_f-t_i)}|a\rangle\,, \end{aligned} \tag{3.2}$$

is the amplitude for the process in which the initial state $|a\rangle$ evolves into the final state $|b\rangle$. In the limit $t_f - t_i \to \infty$, the operator $e^{-iH(t_f-t_i)}$ is called the S–matrix. Therefore S is an operator that maps an initial state to a final state

$$|a\rangle \to S|a\rangle\,, \tag{3.3}$$

an the scattering amplitudes are given by its matrix elements, $\langle b|S|a\rangle$. Observe that

$$\langle a| \to \langle a|S^\dagger\,, \tag{3.4}$$

$$\langle a|a\rangle = 1 \to \langle a|S^\dagger S|a\rangle = 1\,, \tag{3.5}$$

so that $SS^\dagger = S^\dagger S = 1$.

More rigorously,if $\langle a|a\rangle = 1$, and $|n\rangle$ is a complete set of states, the probability that $|a\rangle$ evolves into $|n\rangle$, summed over all $|n\rangle$, must be 1,

$$\sum_n |\langle n|S|a\rangle|^2 = 1. \tag{3.6}$$

On the other hand we can write

$$\begin{aligned}\sum_n |\langle n|S|a\rangle|^2 =& \sum_n \langle a|S^\dagger|n\rangle\langle n|S|a\rangle \\ =& \langle a|S^\dagger \left(\sum_n |n\rangle\langle n|\right) S|a\rangle \\ =& \langle a|S^\dagger S|a\rangle \\ =& 1\,,\end{aligned} \tag{3.7}$$

and we conclude that $SS^\dagger = S^\dagger S = 1$. The unitarity of the S–matrix express the conservation of probability. It is also convenient to define the T matrix, separating the identity operator,

$$S = 1 + iT \tag{3.8}$$

Consider a generic S–matrix element

$$\langle \mathbf{p}_1 \ldots \mathbf{p}_n|S|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle \tag{3.9}$$

For notational simplicity the states are just labeled by their momenta, but all our considerations can be generalized to the case in which the spin is taken into account. We have also defined the operator T from $S = 1 + iT$. We assume that none of the initial momenta $\mathbf{p}_j$ coincides a final momentum $\mathbf{k}_i$. This eliminates processes in which one of the particles behaves as a "spectator" and does not interact with the other particles. In the language of Feynman diagrams to will be explained later, this means that we can restrict to connected diagrams. Therefore, if we restrict to the situation in which no initial and final momenta coincide, the matrix element of the identity operator between these states vanishes, and we need actually to compute the matrix element of iT

$$\langle \mathbf{p}_1 \dots \mathbf{p}_n | iT | \mathbf{k}_1 \dots \mathbf{k}_n \rangle \tag{3.10}$$

In explicit calculations there will be an overall factor Dirac delta imposing energy–momentum conservation. In order not to write explicitly the Dirac delta each time we compute a matrix element of iT, it is convenient to define a matrix element M_{fi} from The matrix element,

$$\langle \mathbf{p}_1 \dots \mathbf{p}_n | iT | \mathbf{k}_1 \dots \mathbf{k}_n \rangle = (2\pi)^4 \delta^{(4)} \left(\sum_j p_j - \sum_j k_j \right) iM_{fi} \,. \tag{3.11}$$

The labels i, f refer to the initial and final states. Explicitly

$$M_{fi} = M(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{k}_1, \dots, \mathbf{k}_n) \,. \tag{3.12}$$

More generally, the initial and final states are labeled also by the spin states of the initial and final particles.

So, instead of S or T, the quantity to be calculated is M_{fi}, but this need first to be relativistically normalized, in which case it will be denoted as $\mathcal{M}_{fi}$.

3.2 Relativistic and no relativistic normalizations

We first consider a system in a cubic box with spatial volume $V = L^3$. At the end of the computation V will be sent to infinity. It is sometimes convenient to put the system into a box of size L, so that the total volume $V = L^3$ is finite. This procedure regularizes divergences coming from the infinite-volume limit or, equivalently, from the small momentum region, and is an example of an infrared cutoff. In a finite box of size L, imposing periodic boundary conditions on the fields, the momenta take the discrete values $\mathbf{p} = 2\pi\mathbf{n}/L$ with $\mathbf{n} = (n_x, n_y, n_z)$ a vector with integer components. In non-relativistic quantum mechanics a one-particle state with momentum $\mathbf{p}$ in the coordinate representation is given by a plane wave

$$\psi_{\mathbf{p}}(\mathbf{x}) = Ce^{i\mathbf{p}\cdot\mathbf{x}} \,, \tag{3.13}$$

and the normalization constant is fixed by the condition that there is one particle in the volume V,

$$\begin{aligned} 1 = \int_V d^3x \, |\psi_{\mathbf{p}}(\mathbf{x})|^2 = & \int_V d^3x \, \psi^*_{\mathbf{p}}(\mathbf{x}) \psi_{\mathbf{p}}(\mathbf{x}) \\ =& |C|^2 \int_V d^3x \\ =& |C|^2 V \,, \end{aligned} \tag{3.14}$$

and

$$\psi_{\mathbf{p}}(\mathbf{x}) = \frac{1}{\sqrt{V}} e^{i\mathbf{p}\cdot\mathbf{x}} . \tag{3.15}$$

Wave functions with different momenta are orthogonal, and therefore

$$\int_V d^3x \, \psi^*_{\mathbf{p}_1}(\mathbf{x}) \psi_{\mathbf{p}_2}(\mathbf{x}) = \delta_{\mathbf{p}_1,\mathbf{p}_2} \tag{3.16}$$

Writing $\psi_{\mathbf{p}}(\mathbf{x}) = \langle \mathbf{x} | \mathbf{p} \rangle$ and using the completeness relation $\int_V d^3x |\mathbf{x}\rangle\langle\mathbf{x}| = 1$, we can write this as

$$\begin{aligned} \langle \mathbf{p}_1 | \mathbf{p}_2 \rangle^{\mathrm{NR}} =& \langle \mathbf{p}_1 | \int_V d^3x |\mathbf{x}\rangle\langle\mathbf{x}|\mathbf{p}_2\rangle \\ =& \int_V d^3x \langle \mathbf{p}_1 | \mathbf{x}\rangle\langle\mathbf{x}|\mathbf{p}_2\rangle \\ =& \int_V d^3x \, \psi^*_{\mathbf{p}_1}(\mathbf{x}) \psi_{\mathbf{p}_2}(\mathbf{x}) \\ =& \, \delta_{\mathbf{p}_1,\mathbf{p}_2} . \end{aligned} \tag{3.17}$$

The superscript NR reminds us that the states have been normalized according to the conventions of non-relativistic quantum mechanics.

In relativistic QFT this normalization is not the most convenient, because the spatial volume V is not relativistically invariant, and therefore the condition "one-particle per volume V" is not invariant. A more convenient Lorentz invariant form was introduced in eq. (2.20)

$$\langle \mathbf{p}_1 | \mathbf{p}_2 \rangle^{\mathrm{R}} = 2E_{\mathbf{p}_1} V \delta_{\mathbf{p}_1,\mathbf{p}_2} \tag{3.18}$$

Therefore the difference between the relativistic and non-relativistic normalization of the one-particle states is, comparing eqs. (3.17) and (3.18)

$$|\mathbf{p}\rangle^{\mathrm{R}} = (2E_{\mathbf{p}} V)^{1/2} \, |\mathbf{p}\rangle^{\mathrm{NR}} \tag{3.19}$$

and of course for a multiparticle state

$$|\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle^{\mathrm{R}} = \left[\prod_{i=1}^{n} (2E_{\mathbf{p}} V)^{1/2} \right] |\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle^{\mathrm{NR}} \tag{3.20}$$

We denote by M_{fi}, defined in eq. (3.11), the scattering amplitude between the initial state with momenta $\mathbf{q}_1, \ldots, \mathbf{q}_n$ and the final state with momenta $\mathbf{p}_1, \ldots, \mathbf{p}_n$, with non-relativistic normalization of the states, and by $\mathcal{M}_{fi}$ the same matrix element with relativistic normalization of the states. Then from eq. (3.11)

$$
\begin{aligned}
(2\pi)^4\delta^{(4)}\left(\sum_i p_i - \sum_i k_i\right) i\mathcal{M}_{fi} =& \langle \mathbf{p}_1 \dots \mathbf{p}_n | iT | \mathbf{k}_1 \dots \mathbf{k}_n \rangle^{\mathrm{R}} \\
=& \prod_{i=1}^{n} (2E_{\mathbf{p}_i}V)^{1/2} \prod_{j=1}^{n} \left(2E_{\mathbf{k}_j}V\right)^{1/2} \langle \mathbf{p}_1 \dots \mathbf{p}_n | iT | \mathbf{k}_1 \dots \mathbf{k}_n \rangle^{\mathrm{NR}} \\
=& \prod_{i=1}^{n} (2E_{\mathbf{p}_i}V)^{1/2} \prod_{j=1}^{n} \left(2E_{\mathbf{k}_j}V\right)^{1/2} (2\pi)^4\delta^{(4)}\left(\sum_i p_i - \sum_i k_i\right) iM_{fi}
\end{aligned}
\tag{3.21}
$$

Therefore

$$
M_{fi} = \prod_{i=1}^{n} (2E_{\mathbf{p}_i}V)^{-1/2} \prod_{j=1}^{n} \left(2E_{\mathbf{k}_j}V\right)^{-1/2} \mathcal{M}_{fi} \tag{3.22}
$$

3.3 Decay Rates

Consider the matrix element of iT in (3.11)

$$
\langle \mathbf{p} | iT | \mathbf{k}_1 \dots \mathbf{k}_n \rangle^{\mathrm{NR}} = (2\pi)^4\delta^{(4)}\left(p - \sum_i k_i\right) iM_{fi} \tag{3.23}
$$

where the initial state is a single particle of momentum p and mass M, while the final state is given by n particles of momenta k_i and masses m_i, $i = 1, \dots, n$. We are therefore considering a decay process.

The relativistic matrix element is therefore

$$
\langle \mathbf{p} | iT | \mathbf{k}_1 \dots \mathbf{k}_n \rangle = (2\pi)^4\delta^{(4)}\left(p - \sum_i k_i\right) \prod_{i=1}^{n} (2E_{\mathbf{p}_i}V)^{-1/2} \prod_{j=1}^{n} \left(2E_{\mathbf{k}_j}V\right)^{-1/2} i\mathcal{M}_{fi} \tag{3.24}
$$

Assume for the moment that all particles are indistinguishable. The rules of quantum mechanics tell us that the probability of this process is obtained by taking the square module of the amplitude

$$
\left|\langle \mathbf{p} | iT | \mathbf{k}_1 \dots \mathbf{k}_n \rangle^{\mathrm{NR}}\right|^2 = \left|(2\pi)^4\delta^{(4)}\left(p - \sum_j k_j\right) iM_{fi}\right|^2 \tag{3.25}
$$

and we are confronted with the square of the delta function. To compute it, we recall that we are working in a finite spatial volume and, from eq. (2.10)

$$(2\pi)^3\delta^{(3)}(0) = V \tag{3.26}$$

Similarly we regularize also the time interval, saying that the time runs from $-T/2$ to $T/2$ so that

$$(2\pi)^4\delta^{(4)}(0) = VT \tag{3.27}$$

Then

$$\begin{aligned} \left|\langle\mathbf{p}|iT|\mathbf{k}_1\ldots\mathbf{k}_n\rangle^{\mathrm{NR}}\right|^2 =& \left|(2\pi)^4\delta^{(4)}\left(p-\sum_j k_i\right)iM_{fi}\right|^2 \\ =&(2\pi)^4\delta^{(4)}\left(p-\sum_i k_j\right)VTM_{fi} \end{aligned} \tag{3.28}$$

Moreover we must sum over all final states. In the discrete limit

Since we are working in a finite volume V, the sum over all final states corresponds to the sum over the possible discrete values of the momenta of the final particles

$$\mathbf{k}_j = \frac{2\pi\mathbf{n}_j}{L}, \qquad \begin{cases} n_j^x = & -\infty,\ldots,-1,0,1,\ldots\infty \\ n_j^y = & -\infty,\ldots,-1,0,1,\ldots\infty \\ n_j^z = & -\infty,\ldots,-1,0,1,\ldots\infty \end{cases} \tag{3.29}$$

$$\sum_{\mathbf{k}_j} = \sum_{n_j^x}\sum_{n_j^y}\sum_{n_j^z} \tag{3.30}$$

In the large-volume limit for each particle we can write, using eq. (2.3)

$$\sum_{\mathbf{k}_j} \to \frac{V}{(2\pi)^3}\int d^3k_j\,, \tag{3.31}$$

The decay probability in (3.28) can be written as

$$\omega = \sum_{\mathbf{k}_1} \dots \sum_{\mathbf{k}_n} \left| \langle \mathbf{p} | iT | \mathbf{k}_1 \dots \mathbf{k}_n \rangle^{\mathrm{NR}} \right|^2 = \sum_{\mathbf{k}_1} \dots \sum_{\mathbf{k}_n} (2\pi)^4 \delta^{(4)} \left(p - \sum_i k_j \right) VT \left| M_{fi} \right|^2$$
$$= \int \dots \int \frac{V d^3 k_1}{(2\pi)^3} \dots \frac{V d^3 k_n}{(2\pi)^3} (2\pi)^4 \delta^{(4)} \left(p - \sum_i k_j \right) VT \left| M_{fi} \right|^2$$
$$= \int \dots \int (2\pi)^4 \delta^{(4)} \left(p - \sum_i k_j \right) VT \left| M_{fi} \right|^2 \prod_{j=1}^{n} \frac{V d^3 k_j}{(2\pi)^3} \,. \quad (3.32)$$

By using eq. (3.22) we have

$$\omega = \int \dots \int (2\pi)^4 \delta^{(4)} \left(p - \sum_i k_j \right) VT \left| \mathcal{M}_{fi} \right|^2 \prod_{j=1}^{n} \frac{V d^3 k_j}{(2\pi)^3} \left((2E_{\mathbf{p}} V)^{-1/2} \prod_{j=1}^{n} \left(2E_{\mathbf{k}_j} V \right)^{-1/2} \right)^2$$
$$= \int \dots \int (2\pi)^4 \delta^{(4)} \left(p - \sum_i k_j \right) T \left| \mathcal{M}_{fi} \right|^2 \frac{1}{2E_{\mathbf{p}}} \prod_{j=1}^{n} \frac{d^3 k_j}{(2\pi)^3 2E_{\mathbf{k}_j}}$$
$$= \int \dots \int (2\pi)^4 \delta^{(4)} \left(p - \sum_i k_j \right) \left(\int dt \right) \left| \mathcal{M}_{fi} \right|^2 \frac{1}{2E_{\mathbf{p}}} \prod_{j=1}^{n} \frac{d^3 k_j}{(2\pi)^3 2E_{\mathbf{k}_j}}$$
$$= \int \dots \int (2\pi)^4 \delta^{(4)} \left(p - \sum_i k_j \right) \frac{1}{2E_{\mathbf{p}}} \left| \mathcal{M}_{fi} \right|^2 dt \prod_{j=1}^{n} \frac{d^3 k_j}{(2\pi)^3 2E_{\mathbf{k}_j}} \quad (3.33)$$

Finally we define the *decay rate* $d\Gamma$ as the decay probability in which in the final state the j–th particle has momentum between k_j and $k_j + dk_j$ per unit time

$$d\Gamma \equiv \frac{d\omega}{dt} = (2\pi)^4 \delta^{(4)} \left(p - \sum_j k_j \right) \frac{1}{2E_{\mathbf{p}}} \left| \mathcal{M}_{fi} \right|^2 \prod_{j=1}^{n} \frac{d^3 k_j}{(2\pi)^3 2E_{\mathbf{k}_j}}$$
$$= \frac{(2\pi)^4}{2E_{\mathbf{p}}} \left| \mathcal{M}_{fi} \right|^2 d\Phi^{(n)}(p; k_1, k_2, \dots, k_n) \quad (3.34)$$

where

$$d\Phi^{(n)}(p; k_1, k_2, \dots, k_n) = \delta^{(4)} \left(p - \sum_j k_j \right) \prod_{j=1}^{n} \frac{d^3 k_j}{(2\pi)^3 2E_{\mathbf{k}_j}} \,. \quad (3.35)$$

If the decaying particle, of mass M is initially at rest, then $E_{\mathbf{p}} = M$, and the differential decay width in the center of mass frame

$$d\Gamma = \frac{(2\pi)^4}{2M}\left|\mathcal{M}_{fi}\right|^2 d\Phi^{(n)}(p; k_1, k_2, \dots, k_n) \tag{3.36}$$

3.4 Cross Section

Consider the collission of a cloud of particles of density n_1^0 moving to the right with velocity v_1^0 toward another cloud of particles in rest of density n_2^0. The differential of number of collisions is

$$\begin{aligned} dN \propto & |\mathbf{v}_1^0| n_1^0 n_2^0 \, dV \, dt \\ = & \sigma |\mathbf{v}_1^0| n_1^0 n_2^0 \, dV \, dt \end{aligned} \tag{3.37}$$

Hence, σ have units of area, and corresponds to the cross section.

The Lorentz invariant expression for velocities and particle densities is

$$|\mathbf{v}_1|^0 n_1^0 n_2^0 \to n_1 n_2 \sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - |\mathbf{v}_1 \times \mathbf{v}_2|} \tag{3.38}$$

Therefore

$$\begin{aligned} dN = & \sigma \sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - |\mathbf{v}_1 \times \mathbf{v}_2|} n_1 n_2 \, dV \, dt \\ = & \sigma \sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - |\mathbf{v}_1 \times \mathbf{v}_2|} \frac{1}{V} (n_1 V)(n_2 \, dV) \, dt \\ = & \sigma \sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - |\mathbf{v}_1 \times \mathbf{v}_2|} \frac{1}{V} N_1 (n_2 \, dV) \, dt \,. \end{aligned} \tag{3.39}$$

Integrate it out

$$N = \sigma \sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - |\mathbf{v}_1 \times \mathbf{v}_2|} \frac{1}{V} N_1 N_2 T \tag{3.40}$$

The probability for the collision to happens is

$$\frac{N}{N_1 N_2} = \frac{\sigma T}{V} \sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - |\mathbf{v}_1 \times \mathbf{v}_2|} \tag{3.41}$$

As in equation (3.32), the collision probability is

$$\frac{N}{N_1 N_2} = \int \dots \int (2\pi)^4 \delta^{(4)} \left(\sum_i p_i - \sum_f p_f \right) VT \left|M_{fi}\right|^2 \prod_f \frac{V d^3 p_f}{(2\pi)^3} \,. \tag{3.42}$$

Replacing back in eq. (3.41), we have that the differential cross section is

$$d\sigma = \frac{V}{T\sqrt{(\mathbf{v}_1-\mathbf{v}_2)^2-|\mathbf{v}_1\times\mathbf{v}_2|}}(2\pi)^4\delta^{(4)}\left(\sum_i p_i - \sum_f p_f\right) VT\,|M_{fi}|^2 \prod_f \frac{V d^3p_f}{(2\pi)^3}\,. \tag{3.43}$$

Defining

$$I = E_1E_2\sqrt{(\mathbf{v}_1-\mathbf{v}_2)^2-|\mathbf{v}_1\times\mathbf{v}_2|} \tag{3.44}$$

and using eq. (3.22) for the relativistic scattering amplitude

$$\begin{aligned} d\sigma =& \frac{VE_1E_2}{TI}(2\pi)^4\delta^{(4)}\left(\sum_i p_i - \sum_f p_f\right) VT\,|M_{fi}|^2 \prod_f \frac{V d^3p_f}{(2\pi)^3} \\ =& \frac{V^2E_1E_2}{TI}(2\pi)^4\delta^{(4)}\left(\sum_i p_i - \sum_f p_f\right) |M_{fi}|^2 \prod_f \frac{V d^3p_f}{(2\pi)^3} \\ =& \frac{V^2E_1E_2}{TI}(2\pi)^4\delta^{(4)}\left(\sum_i p_i - \sum_f p_f\right) \frac{1}{2E_1V}\frac{1}{2E_2V}\prod_f \frac{1}{2E_{\mathbf{p}_f}V}\,|\mathcal{M}_{fi}|^2 \prod_f \frac{V d^3p_f}{(2\pi)^3}\,. \end{aligned} \tag{3.45}$$

In this way, when the initial state in the S–matrix contains two particles

$$d\sigma = (2\pi)^4\delta^{(4)}\left(\sum_{i=1,2} p_i - \sum_f p_f\right)\frac{1}{4I}\,|\mathcal{M}_{fi}|^2 \prod_f \frac{d^3p_f}{(2\pi)^3 2E_{\mathbf{p}_f}} \tag{3.46}$$

where

$$I = \sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2} \tag{3.47}$$

Defining

$$v_{\text{rel}} = \frac{I}{E_1E_2} \tag{3.48}$$

In general

$$\begin{aligned} I =& \sqrt{(E_1E_2 - \mathbf{p}_1\cdot\mathbf{p}_2)^2 - m_1^2m_2^2} \\ =& \sqrt{E_1^2E_2^2 + (\mathbf{p}_1\cdot\mathbf{p}_2)^2 - 2E_1E_2\mathbf{p}_1.\mathbf{p}_2 - m_1^2m_2^2} \end{aligned} \tag{3.49}$$

Since

$$\begin{aligned} m_1^2 m_2^2 =&(E_1^2-\mathbf{p}_1^2)(E_2^2-\mathbf{p}_2^2)\\ =&(E_1^2E_2^2-\mathbf{p}_1^2E_2^2-E_1^2\mathbf{p}_2^2+\mathbf{p}_1^2\mathbf{p}_2^2) \end{aligned} \tag{3.50}$$

$$I=\sqrt{\mathbf{p}_1^2E_2^2-2E_1E_2\mathbf{p}_1\cdot\mathbf{p}_2+E_1^2\mathbf{p}_2^2+(\mathbf{p}_1\cdot\mathbf{p}_2)^2-\mathbf{p}_1^2\mathbf{p}_2^2} \tag{3.51}$$

If

$$(\mathbf{p}_1\cdot\mathbf{p}_2)^2-\mathbf{p}_1^2\mathbf{p}_2^2=0 \tag{3.52}$$

that implies that $\mathbf{p}_1$ and $\mathbf{p}_2$ are colineals,

$$\begin{aligned} I =&\sqrt{\mathbf{p}_1^2E_2^2-2E_1E_2\mathbf{p}_1\cdot\mathbf{p}_2+E_1^2\mathbf{p}_2^2}\\ =&\sqrt{(\mathbf{p}_1E_2-\mathbf{p}_2E_1)^2}\\ =&|\mathbf{p}_1E_2-\mathbf{p}_2E_1| \end{aligned} \tag{3.53}$$

$$v_{\text{rel}}=\frac{I}{E_1E_2}=\left|\frac{\mathbf{p}_1}{E_1}-\frac{\mathbf{p}_2}{E_2}\right| \tag{3.54}$$

$$d\sigma=(2\pi)^4\delta^{(4)}\left(\sum_i p_i-\sum_f p_f\right)\frac{1}{4v_{\text{rel}}E_1E_2}\left|\mathcal{M}_{fi}\right|^2\prod_f\frac{d^3p_f}{(2\pi)^3 2E_f} \tag{3.55}$$

3.4.1 2–to–2 cross section

The the 2–to–2 cross section is

$$\begin{aligned} d\sigma =&(2\pi)^4\delta^{(4)}\left(p_1+p_2-p_1'-p_2'\right)\frac{1}{v_{\text{rel}}}\frac{1}{4E_1E_2}\overline{|\mathcal{M}_{fi}|^2}\frac{d^3p_1'}{(2\pi)^3 2E_1'}\frac{d^3p_2'}{(2\pi)^3 2E_2'}\\ =&2^4\delta^{(4)}\left(p_1+p_2-p_1'-p_2'\right)\frac{1}{v_{\text{rel}}}\frac{1}{2^2E_1E_2}\overline{|\mathcal{M}_{fi}|^2}\frac{d^3p_1'}{2^3 2E_1'}\frac{d^3p_2'}{2^3\pi^2 2E_2'}\\ =&\frac{1}{64\pi^2E_1E_2v_{\text{rel}}}\overline{|\mathcal{M}_{fi}|^2}\delta^{(4)}\left(p_1+p_2-p_1'-p_2'\right)\frac{d^3p_1'}{E_1'}\frac{d^3p_2'}{E_2'}\\ =&\frac{1}{64\pi^2E_1E_2v_{\text{rel}}}\overline{|\mathcal{M}_{fi}|^2}\left[4(2\pi)^2 d\Phi^{(2)}(p_1,p_2;p_1',p_2')\right] \end{aligned} \tag{3.56}$$

where, as in eq. (3.35)

$$d\Phi^{(2)}(p_1,p_2;p_1',p_2') = \frac{1}{4(2\pi)^2}\delta^{(4)}\left(p_1+p_2-p_1'-p_2'\right)\frac{d^3p_1'}{E_1'}\frac{d^3p_2'}{E_2'} \tag{3.57}$$

We now will find an expression for cross section in the center of mass frame (CM)

The center of mass (CM) frame is defined by the condition

$$\mathbf{p}_1+\mathbf{p}_2=0 \tag{3.58}$$

The δ–function in Eq. (5.82)

$$\delta^{(4)}(p+p_2-p_1'-p_2') = \delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2')\delta(E_1+E_2-E_1'-E_2') \tag{3.59}$$

In the CM frame

$$\delta^{(4)}(p+p_2-p_1'-p_2') = \delta^{(3)}(\mathbf{p}_1'+\mathbf{p}_2')\delta(E_1+E_2-E_1'-E_2') \tag{3.60}$$

$\mathcal{M}_{fi}$ in integration does not depend on $|\mathbf{p}_1'|$ or $|\mathbf{p}_2'|$ as the final momentum is fixed by the initial momentum whenever the final states have only two particles. In this way the integration on p_2' can be evaluated directly for $d\Phi^{(2)}$. Replacing back in Eq. (3.56)

$$\begin{aligned} d\Phi^{(2)} =& \frac{1}{4(2\pi)^2}\delta^{(3)}(\mathbf{p}_1'+\mathbf{p}_2')\delta(E_1+E_2-E_1'-E_2')\frac{d^3p_1'}{E_1'}\frac{d^3p_2'}{E_2'}\\ =& \frac{1}{4(2\pi)^2}\delta(E_1+E_2-E_1'-E_2')\frac{d^3p_1'}{E_1'}\int\delta^{(3)}(\mathbf{p}_1'+\mathbf{p}_2')\frac{d^3p_2'}{E_2'}\\ =& \frac{1}{4(2\pi)^2}\delta(E_1+E_2-E_1'-E_2')\frac{d^3p_1'}{E_1'E_2'} \end{aligned} \tag{3.61}$$

$$d\Phi^{(2)} = \frac{1}{4(2\pi)^2}\delta(E_1+E_2-E_1'-E_2')\frac{{\mathbf{p}_1'}^2 d|\mathbf{p}_1'|d\Omega}{E_1'E_2'} \tag{3.62}$$

As

$$|\mathbf{p}_1'| = \sqrt{{E_1'}^2-{m_1}^2} \tag{3.63}$$

$$\begin{aligned} \frac{d|\mathbf{p}_1'|}{dE_1'} =& \frac{2E_1'}{2\sqrt{{E_1'}^2-{m_1}^2}}\\ =& \frac{E_1'}{|\mathbf{p}_1'|} \end{aligned} \tag{3.64}$$

In this way, we can write, in general

$$|\mathbf{p}|\, d|\mathbf{p}| = E\, dE \tag{3.65}$$

and

$$\begin{aligned} d\Phi^{(2)} &= \frac{1}{4(2\pi)^2}\delta(E_1+E_2-E_1'-E_2')\frac{|\mathbf{p}_1'|E_1' dE_1'}{E_1'E_2'}d\Omega \\ &= \frac{1}{4(2\pi)^2}\delta(E_1+E_2-E_1'-E_2')\frac{|\mathbf{p}_1'|dE_1'}{E_2'}d\Omega \end{aligned} \tag{3.66}$$

From the δ–function in Eq. (3.59) we have that in the CM frame

$$\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2' = 0 \overset{\text{CM}}{\Rightarrow} \begin{cases} \mathbf{p}_1 = -\mathbf{p}_2 \\ \mathbf{p}_1' = -\mathbf{p}_2' \end{cases} \tag{3.67}$$

Squaring the first expression, and taking into account that

$$\mathbf{p}_1' = \sqrt{{E_1'}^2 - {m_1'}^2} \tag{3.68}$$

we have

$$\begin{aligned} {\mathbf{p}_1'}^2 =& {\mathbf{p}_2'}^2 \\ {E_1'}^2 - {m_1'}^2 =& {E_2'}^2 - {m_2'}^2\,, \end{aligned} \tag{3.69}$$

$$E_2' = \sqrt{{E_1'}^2 - {m_1'}^2 + {m_2'}^2} \tag{3.70}$$

In this way we can express E_2' in terms of E_1' in Eq. (3.66). Moreover, we can define the center of mass energy as

$$\sqrt{s} = E_1 + E_2 \tag{3.71}$$

Using The energy part of δ–function in Eq. (3.59) can be written as

$$\delta\left(\sqrt{s} - E_1' - \sqrt{{E_1'}^2 - {m_1'}^2 + {m_2'}^2}\right) \tag{3.72}$$

As established before, $\mathcal{M}_{fi}$ in this case in independent of $|\mathbf{p}_1'|$, and the integration on E_1' can be done directly only for $d\Phi^{(2)}$. The integral is easily performed using the identity

$$\delta\left(f(z)\right) = \sum_n \frac{\delta(z-z_n)}{|f'(z_n)|} \tag{3.73}$$

where z_n are the zeroes of $f(z)$. In this case, this δ–function is a function of the integration variable E'_1, with only one zerp

$$\delta\left(f(x)\right) = \frac{\delta(x-x_0)}{|f'(x_0)|} \tag{3.74}$$

where

$$f(x) = \sqrt{s} - x - \sqrt{x^2 - {m'_1}^2 + {m'_2}^2} \tag{3.75}$$

Therefore

$$\begin{aligned} d\Phi^{(2)} =& \frac{1}{4(2\pi)^2} d\Omega \int \frac{\delta(x-x_0)}{|f'(x_0)|} \frac{|\mathbf{p}'_1(x)|}{E'_2(x)} dx \\ =& \frac{1}{4(2\pi)^2} d\Omega \frac{1}{|f'(x_0)|} \frac{|\mathbf{p}'_1(x_0)|}{E'_2(x_0)} \end{aligned} \tag{3.76}$$

where from Eqs. (3.68), (3.70),

$$\mathbf{p}'_1(x_0) = \sqrt{x_0^2 - {m'_1}^2} \qquad\qquad E'_2(x_0) = \sqrt{x_0^2 - {m'_1}^2 + {m'_2}^2} \tag{3.77}$$

The zero is obtained from

$$\begin{aligned} &\sqrt{s} - x_0 - \sqrt{x_0^2 - {m'_1}^2 + {m'_2}^2} = 0 \\ &s - 2\sqrt{s}\, x_0 + x_0^2 = x_0^2 - {m'_1}^2 + {m'_2}^2 \\ &s - 2\sqrt{s}\, x_0 = -{m'_1}^2 + {m'_2}^2 \end{aligned} \tag{3.78}$$

with solution

$$x_0 = \frac{s + {m'_1}^2 - {m'_2}^2}{2\sqrt{s}} \tag{3.79}$$

As (See `deltaxn.nb` for additional details)

$$f'(x) = -\frac{x}{\sqrt{x^2 - {m'_1}^2 + {m'_2}^2}} - 1 \tag{3.80}$$

we have

$$
\begin{aligned}
f'(x_0) =& -\frac{{m_1'}^2 - {m_2'}^2 + s}{\sqrt{s}\sqrt{\frac{\left(-{m_1'}^2+{m_2'}^2+s\right)^2}{s}}} - 1 \\
=&\frac{-{m_1'}^2 + {m_2'}^2 - s}{-{m_1'}^2 + {m_2'}^2 + s} - 1 \\
=&\frac{-{m_1'}^2 + {m_2'}^2 - s + {m_1'}^2 - {m_2'}^2 - s}{-{m_1'}^2 + {m_2'}^2 + s} \\
=&\frac{-2s}{s + {m_2'}^2 - {m_1'}^2}\,,
\end{aligned}
\tag{3.81}
$$

and

$$
\delta(f(E_1')) = \delta(E_1' - x_0)\left(\frac{s + {m_2'}^2 - {m_1'}^2}{2s}\right)
\tag{3.82}
$$

Replacing the expression for x_0 in (3.79) into Eq. (3.77) we have (See `deltaxn.nb` for additional details)

$$
\begin{aligned}
\mathbf{p}_1'(x_0) =&\frac{\sqrt{[s - ({m_1'}^2 + {m_2'}^2)][s - ({m_1'}^2 - {m_2'}^2)]}}{2\sqrt{s}} \\
E_2'(x_0) =&\frac{s - {m_1'}^2 + {m_2'}^2}{2\sqrt{s}}
\end{aligned}
\tag{3.83}
$$

Replacing Eqs. (3.81), and (3.83) in Eq. (3.76) we have

$$
\begin{aligned}
d\Phi^{(2)} =& \frac{1}{4(2\pi)^2}d\Omega\frac{1}{|f'(x_0)|}\frac{\sqrt{x_0^2 - {m_1'}^2}}{\sqrt{x_0^2 - {m_1'}^2 + {m_2'}^2}} \\
=& \frac{1}{4(2\pi)^2}d\Omega\left(\frac{s - {m_1'}^2 + {m_2'}^2}{2s}\right)\frac{\sqrt{[s - ({m_1'}^2 + {m_2'}^2)][s - ({m_1'}^2 - {m_2'}^2)]}}{s - {m_1'}^2 + {m_2'}^2} \\
=& \frac{1}{4(2\pi)^2}d\Omega\frac{\sqrt{[s - ({m_1'}^2 + {m_2'}^2)][s - ({m_1'}^2 - {m_2'}^2)]}}{2s}
\end{aligned}
\tag{3.84}
$$

To further evaluate Eq. (3.56), we need to express v_{rel} and E_1E_2 in terms of s and the masses. Concerning v_{rel}, from Eq. (3.54), evaluated in CM frame

$$\begin{aligned} E_1E_2v_{\text{rel}} =& E_1E_2\left|\frac{\mathbf{p}_1}{E_1}-\frac{\mathbf{p}_2}{E_2}\right| \\ =& E_1E_2\left|\frac{\mathbf{p}_1}{E_1}+\frac{\mathbf{p}_1}{E_2}\right| \\ =& \left|\mathbf{p}_1\right|(E_1+E_2) \\ =& \left|\mathbf{p}_1\right|\sqrt{s} \end{aligned} \tag{3.85}$$

Replacing back Eqs. (3.84), and (3.85) into Eq. (3.56), we have

$$d\sigma = \frac{1}{64\pi^2 E_1E_2v_{\text{rel}}}\overline{|\mathcal{M}_{fi}|^2}\left[4(2\pi)^2 d\Phi^{(2)}\right] \tag{3.86}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_1E_2v_{\text{rel}}}\overline{|\mathcal{M}_{fi}|^2}\frac{\sqrt{[s-({m'_1}^2+{m'_2}^2)][s-({m'_1}^2-{m'_2}^2)]}}{2s} \tag{3.87}$$

By using Eq. (3.85)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 |\mathbf{p}_1|\sqrt{s}}\overline{|\mathcal{M}_{fi}|^2}\frac{\sqrt{[s-({m'_1}^2+{m'_2}^2)][s-({m'_1}^2-{m'_2}^2)]}}{2s} \tag{3.88}$$

In the CM frame

$$\begin{aligned} \sqrt{s} =& E_1+E_2 \\ =& \sqrt{\mathbf{p}_1^2+m_1^2}+\sqrt{\mathbf{p}_2^2+m_2^2} \\ =& \sqrt{\mathbf{p}_1^2+m_1^2}+\sqrt{\mathbf{p}_1^2+m_2^2} \end{aligned} \tag{3.89}$$

$$\begin{aligned} s =& 2\mathbf{p}_1^2+m_1^2+m_2^2+2\sqrt{\mathbf{p}_1^4+(m_1^2+m_2^2)\mathbf{p}_1^2+m_1^2m_2^2} \\ s-(2\mathbf{p}_1^2+m_1^2+m_2^2) =& 2\sqrt{\mathbf{p}_1^4+(m_1^2+m_2^2)\mathbf{p}_1^2+m_1^2m_2^2} \end{aligned} \tag{3.90}$$

$$
\begin{aligned}
s^2-2s(2\mathbf{p}_1^2+m_1^2+m_2^2)+[2\mathbf{p}_1^2+(m_1^2+m_2^2)]^2 =&4(\mathbf{p}_1^4+(m_1^2+m_2^2)\mathbf{p}_1^2+m_1^2m_2^2)\\
s^2-2s(2\mathbf{p}_1^2+m_1^2+m_2^2)+4\mathbf{p}_1^4+4\mathbf{p}_1^2(m_1^2+m_2^2)+(m_1^2+m_2^2)^2 =&4(\mathbf{p}_1^4+(m_1^2+m_2^2)\mathbf{p}_1^2+m_1^2m_2^2)\\
-4s\mathbf{p}_1^2+s^2-2s(m_1^2+m_2^2)+(m_1^2+m_2^2)^2 =&4m_1^2m_2^2\\
-4s\mathbf{p}_1^2+s^2-2sm_1^2-2sm_2^2+m_1^4+m_2^4+2m_1^2m_2^2 =&4m_1^2m_2^2\\
-4s\mathbf{p}_1^2+s^2-2sm_1^2-2sm_2^2+m_1^4+m_2^4-2m_1^2m_2^2 =&0
\end{aligned}
\tag{3.91}
$$

$$
\mathbf{p}_1^2=\frac{(s-m_1^2-2m_2m_1-m_2^2)\,(s-m_1^2+2m_2m_1-m_2^2)}{4s} \tag{3.92}
$$

$$
|\mathbf{p}_1|=\frac{\sqrt{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}}{2\sqrt{s}} \tag{3.93}
$$

Replacing Eq. (3.93) back in Eq. (3.85) we have

$$
E_1E_2v_{\rm rel}=\frac{1}{2}\sqrt{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]} \tag{3.94}
$$

Replacing Eqs. (3.94), and (3.84) in Eq. (3.56)

$$
d\sigma=\frac{1}{64\pi^2}\overline{|\mathcal{M}_{fi}|^2}\frac{d\Omega}{2s}(2)\sqrt{\frac{[s-(m_1'+m_2')^2][s-(m_1'-m_2')^2]}{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}} \tag{3.95}
$$

and, finally

$$
\frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2s}\left\{\frac{[s-(m_1'+m_2')^2][s-(m_1'-m_2')^2]}{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}\right\}^{1/2}\overline{|\mathcal{M}|^2} \tag{3.96}
$$

3.5 Backup

Perturbation theory is developed more easily using the Hamiltonian formalism. We therefore consider a general field theory with a Hamiltonian

$$
H=H_0+H_{\rm int} \tag{3.97}
$$

where H_0 is the free Hamiltonian and $H_{\rm int}$ is the interaction term. The interaction term will be considered small. For instance in QED

$$
H_{\rm int}=\int d^3x\,\mathcal{H}_{\rm int}=-\int d^3x\,\mathcal{L}_{\rm int} \tag{3.98}
$$

with

$$\mathcal{L}_{\rm int} = -eA_\mu \overline{\psi}\gamma^\mu \psi \tag{3.99}$$

The smallnes of the interaction follows from the fact that the parameter which turns out to be relevatn for the perturbation expansion is $\alpha = e^2/4\pi \approx 1/137$.

$$\begin{aligned} SS^\dagger =&(1+iT)(1-iT^\dagger) \\ &= 1 + i(T-T^\dagger) + TT^\dagger = 1\,, \end{aligned} \tag{3.100}$$

$$TT^\dagger = -i(T-T^\dagger)\,. \tag{3.101}$$

Inserting a complete set of states we have

$$\begin{aligned} \langle b|TT^\dagger|a\rangle =& -i(\langle b|T|a\rangle - \langle b|T^\dagger|a\rangle) \\ \langle b|T\left(\sum_n |n\rangle\langle n|\right)T^\dagger|a\rangle =& -i\left[\langle b|T|a\rangle - (\langle a|T|b\rangle)^\dagger\right] \\ \sum_n \langle b|T|n\rangle\langle a|T|n\rangle^\dagger =& -i\left(\langle b|T|a\rangle - \langle a|T|b\rangle^*\right) \\ \sum_n T_{bn}T^*_{an} =& -i\left(T_{ba} - T^*_{ab}\right)\,. \end{aligned} \tag{3.102}$$

if $a=b$

$$|T_{an}|^2 = -i\,\mathrm{Im}\,T_{aa}\,. \tag{3.103}$$