Chapter 2

Second quantization

This part is based in some topics of chapters 4-6 of [2].

2.1 Fock space for real scalar fields

We have already seen in Chapter 1 of [1] that the most general solution to the Klein–Gordon equation is

$$\phi(t,\mathbf{x})=\sum_{\mathbf{n}}\frac{1}{\sqrt{2E_{\mathbf{n}}L^3}}\left(a_{\mathbf{n}}e^{-ip_{\mathbf{n}}\cdot x}+a^*_{\mathbf{n}}e^{ip_{\mathbf{n}}\cdot x}\right)\tag{2.1}$$

with $p^0=E_{\mathbf{n}}$. The factor $\sqrt{2E_{\mathbf{n}}}$ is a convenient choice of normalization of the coefficients a_n which guarantees the proper harmonic oscillator Hamiltonian

$$H=\sum_n E_{\mathbf{n}}a^*_{\mathbf{n}}a_{\mathbf{n}}\tag{2.2}$$

In the continuum

$$\left(\frac{2\pi}{L}\right)^3\sum_{\mathbf{n}}=\frac{(2\pi)^3}{V}\sum_{\mathbf{n}}\to\int d^3p\tag{2.3}$$

$$\phi(t,\mathbf{x})=\int d^3p\frac{\sqrt{L}}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}+a^*_{\mathbf{p}}e^{ip\cdot x}\right)\tag{2.4}$$

The basic principle of canonical quantization is to promote the field ϕ and its conjugate momentum to operators, and to impose the equal time commutation relation

$$\begin{aligned}\left[\widehat{\phi}(t,\mathbf{x}),\widehat{\Pi}(t,\mathbf{y})\right] &= \delta^3(\mathbf{x},\mathbf{y}) \\ \left[\widehat{\phi}(t,\mathbf{x}),\widehat{\phi}(t,\mathbf{y})\right] &= \left[\widehat{\Pi}(t,\mathbf{x}),\widehat{\Pi}(t,\mathbf{y})\right] = 0\,.\end{aligned} \tag{2.5}$$

Promoting the real field ϕ to a hermitian operator means to promote $a_{\mathbf{p}}$ to an operator; thus

$$\widehat{\phi}(t,\mathbf{x}) = \int d^3p \frac{\sqrt{L}}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\left(\widehat{a}_{\mathbf{p}}e^{-ip\cdot x} + \widehat{a}^{\dagger}_{\mathbf{p}}e^{ip\cdot x}\right) \tag{2.6}$$

Using these expressions it is easy to verify that, in terms of $\widehat{a}_{\mathbf{p}}$, $\widehat{a}^{\dagger}_{\mathbf{p}}$, the commutation relation (2.5) reads

$$\begin{aligned}\left[\widehat{a}_{\mathbf{p}},\widehat{a}^{\dagger}_{\mathbf{q}}\right] &= \left(\frac{2\pi}{L}\right)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) \\ \left[\widehat{a}_{\mathbf{p}},\widehat{a}_{\mathbf{q}}\right] &= \left[\widehat{a}^{\dagger}_{\mathbf{p}},\widehat{a}^{\dagger}_{\mathbf{q}}\right] = 0\,.\end{aligned} \tag{2.7}$$

Comparing with the commutation relation of harmonic oscillator studied before []

$$\begin{aligned}\left[\widehat{a}_{\mathbf{n}},\widehat{a}^{\dagger}_{\mathbf{m}}\right] &= \delta_{\mathbf{n},\mathbf{m}} \\ \left[\widehat{a}_{\mathbf{n}},\widehat{a}_{\mathbf{m}}\right] &= \left[\widehat{a}^{\dagger}_{\mathbf{n}},\widehat{a}^{\dagger}_{\mathbf{m}}\right] = 0\,,\end{aligned} \tag{2.8}$$

we get that in the continuum limit

$$\left(\frac{L}{2\pi}\right)^3 \delta_{\mathbf{p},\mathbf{q}} \to \delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{2.9}$$

In particular, this implies that

$$(2\pi)^3\delta^{(3)}(\mathbf{p}=0) \to L^3 = V \tag{2.10}$$

$$\delta^3(\mathbf{0}) = \frac{V}{(2\pi)^3}\,. \tag{2.11}$$

This expression can be also obtained from the definition

$$\delta^3(\mathbf{p}) = \lim_{V\to\infty}\left(\frac{1}{(2\pi)^3}\int_V d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}\right)\,, \tag{2.12}$$

before taking the limit to infinity.

Note that the commutation relations for the real scalar field in (2.7) are equivalent to that of a collection of harmonic oscillators, with one oscillator for each value of the momentum $\mathbf{p}$.

We can now construct the Fock space following the standard procedure for the harmonic oscillator: we interpret $\widehat{a}_{\mathbf{p}}$ as destruction operators and $\widehat{a}^\dagger_{\mathbf{p}}$ as creation operators, and we define a vacuum state $|0\rangle$ as the state annihilated by all destruction operators, so for all $\mathbf{p}$

$$\widehat{a}_{\mathbf{p}}|0\rangle = 0\,. \tag{2.13}$$

We normalize the vacuum with $\langle 0|0\rangle = 1$. The generic state of the Fock space is obtained acting on the vacuum with the creation operators,

$$|\mathbf{p}_1,\ldots,\mathbf{p}_n\rangle \equiv (2E_{\mathbf{p_1}})^{1/2}\ldots(2E_{\mathbf{p_n}})^{1/2}\,\widehat{a}^\dagger_{\mathbf{p}_1}\ldots\widehat{a}^\dagger_{\mathbf{p}_n}|0\rangle\,. \tag{2.14}$$

The factors $(2E_{\mathbf{p_1}})^{1/2}$ are a convenient choice of normalization. In particular, the one-particle states are

$$|\mathbf{p}\rangle = (2E_{\mathbf{p}})^{1/2}\,\widehat{a}^\dagger_{\mathbf{p}}|0\rangle\,. \tag{2.15}$$

From the commutations relations and eq. (2.7) we find that

$$\begin{aligned}
\langle\mathbf{p}_1|\mathbf{p}_2\rangle &= (2E_{\mathbf{p_1}})^{1/2}\,(2E_{\mathbf{p_2}})^{1/2}\,\langle 0|\widehat{a}_{\mathbf{p}_1}\widehat{a}^\dagger_{\mathbf{p}_2}|0\rangle \\
&= (2E_{\mathbf{p_1}})^{1/2}\,(2E_{\mathbf{p_2}})^{1/2}\,\langle 0|\left[\widehat{a}_{\mathbf{p}_1},\widehat{a}^\dagger_{\mathbf{p}_2}\right]|0\rangle \\
&= (2E_{\mathbf{p_1}})^{1/2}\,(2E_{\mathbf{p_2}})^{1/2}\left(\frac{2\pi}{L}\right)^3\delta^{(3)}(\mathbf{p}_1-\mathbf{p}_2) \\
&= 2E_{\mathbf{p_1}}\left(\frac{2\pi}{L}\right)^3\delta^{(3)}(\mathbf{p}_1-\mathbf{p}_2)\,.
\end{aligned} \tag{2.16}$$

The factors $(2E_{\mathbf{p}})^{1/2}$ in eq. (2.15) have been chosen so that in the above product the combination $E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q})$ appears, which is Lorentz invariant. To see this perform a boost along z–axis. Since the transverse components of the momentum are no affected we must consider only $E_{\mathbf{p}}\delta(p_z-k_z)$. Use the form of the Lorentz transformation of $E_{\mathbf{p}}, p_z$, together with the property of the Dirac delta $\delta(f(x)) = \delta(x-x_0)/f'(x_0)$.

However, because eq. (2.6) is already in the continuum, the $\sqrt{L}$ factor must not appear. We can reabsorb this in the $\widehat{a}_{\mathbf{p}}, \widehat{a}^\dagger_{\mathbf{p}}$ definition, so that

$$\widehat{\phi}(t,\mathbf{x}) = \int d^3p\frac{1}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\left(\widehat{a}_{\mathbf{p}}e^{-ip\cdot x}+\widehat{a}^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)\,. \tag{2.17}$$

The new commutation relations are

$$\begin{aligned}\left[\widehat{a}_{\mathbf{p}}, \widehat{a}^{\dagger}_{\mathbf{q}}\right] &= (2\pi)^3\, \delta^{(3)}(\mathbf{p}-\mathbf{q}) \\ \left[\widehat{a}_{\mathbf{p}}, \widehat{a}_{\mathbf{q}}\right] &= \left[\widehat{a}^{\dagger}_{\mathbf{p}}, \widehat{a}^{\dagger}_{\mathbf{q}}\right] = 0\,,\end{aligned} \tag{2.18}$$

and eq. (2.16) is now

$$\langle \mathbf{p}_1 | \mathbf{p}_2 \rangle = 2E_{\mathbf{p_1}}\, (2\pi)^3\, \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2)\,. \tag{2.19}$$

Using (2.9) we have in a finite box

$$\begin{aligned}\langle \mathbf{p}_1 | \mathbf{p}_2 \rangle =& 2E_{\mathbf{p_1}} L^3 \delta_{\mathbf{p_1},\mathbf{p_2}} \\ =& 2E_{\mathbf{p_1}} V \delta_{\mathbf{p_1},\mathbf{p_2}}\end{aligned} \tag{2.20}$$

If we define

$$\widehat{\phi}(t,\mathbf{x}) = \int d^3p \frac{1}{\sqrt{(2\pi)^3 2E_{\mathbf{p}}}} \left(\widehat{a}_{\mathbf{p}} e^{-ip\cdot x} + \widehat{a}^{\dagger}_{\mathbf{p}} e^{ip\cdot x} \right)\;. \tag{2.21}$$

Then, the new commutation relations are

$$\begin{aligned}\left[\widehat{a}_{\mathbf{p}}, \widehat{a}^{\dagger}_{\mathbf{q}}\right] &= \delta^{(3)}(\mathbf{p}-\mathbf{q}) \\ \left[\widehat{a}_{\mathbf{p}}, \widehat{a}_{\mathbf{q}}\right] &= \left[\widehat{a}^{\dagger}_{\mathbf{p}}, \widehat{a}^{\dagger}_{\mathbf{q}}\right] = 0\,,\end{aligned} \tag{2.22}$$

It is convinient to define:

$$\widehat{\phi}(x) = \widehat{\phi}_+(x) + \widehat{\phi}_-(x) \tag{2.23}$$

where

$$\begin{aligned}\widehat{\phi}_+(x) &= \int d^3p \frac{1}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \widehat{a}_{\mathbf{p}} e^{-ip\cdot x} \\ \widehat{\phi}_-(x) &= \int d^3p \frac{1}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \widehat{a}^{\dagger}_{\mathbf{p}} e^{ip\cdot x}\,.\end{aligned} \tag{2.24}$$

We define the Fock one-particle state as

$$\begin{aligned}|B(\mathbf{p})\rangle =& A\, \widehat{a}^{\dagger}_{\mathbf{p}} |0\rangle \\ \langle B(\mathbf{p})| =& \langle 0| \widehat{a}_{\mathbf{p}}\, A^*\end{aligned} \tag{2.25}$$

$$\begin{aligned}\langle B(\mathbf{p})|B(\mathbf{p}')\rangle =& A^2\langle 0|\widehat{a}_{\mathbf{p}}\widehat{a}^{\dagger}_{\mathbf{p}'}|0\rangle \\ =& A^2\langle 0|\widehat{a}_{\mathbf{p}}\widehat{a}^{\dagger}_{\mathbf{p}'} - \widehat{a}^{\dagger}_{\mathbf{p}'}\widehat{a}_{\mathbf{p}}|0\rangle \\ =& A^2\langle 0|[\widehat{a}_{\mathbf{p}}, \widehat{a}^{\dagger}_{\mathbf{p}'}]|0\rangle \\ =& A^2(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}') \end{aligned} \tag{2.26}$$

As we fix the normalization as

$$A = \frac{1}{V} \tag{2.27}$$

so that

$$|B(\mathbf{p})\rangle = \frac{1}{\sqrt{V}}\,\widehat{a}^{\dagger}(\mathbf{p})|0\rangle \tag{2.28}$$

2.2 Quantization of Fermions

The solutions to the free Dirac equations are

$$\begin{aligned}\psi_{\text{particle}}(x) =& \frac{1}{\sqrt{2E_pV}}u_s(\mathbf{p})e^{-ip\cdot x} \\ \psi_{\text{antiparticle}}(x) =& \frac{1}{\sqrt{2E_pV}}v_s(\mathbf{p})e^{ip\cdot x} \end{aligned} \tag{2.29}$$

In a similarly way to eq. (2.23), the most general free particle solution to Dirac equation is

$$\widehat{\psi}(x) = \widehat{\psi}_+(x) + \widehat{\psi}_-(x) \tag{2.30}$$

$$\begin{aligned}\widehat{\psi}_+(x) =& \int d^3p \frac{1}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\sum_{s=1,2}\hat{f}_s(\mathbf{p})u_s(\mathbf{p})e^{-ip\cdot x} \\ \widehat{\psi}_-(x) =& \int d^3p \frac{1}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\sum_{s=1,2}\hat{f}^{\dagger}_s(\mathbf{p})v_s(\mathbf{p})e^{ip\cdot x} \end{aligned} \tag{2.31}$$